Ordinary Differential Equations **Review Problems** **Childress, Spring 2002**

Due on or before May 2 in 717 WWH. Additional problems to be assigned April 23

1. Solve the initial value problem

$$y\frac{dy}{dx} + (1+y^2)\sin x = 0,\ y(0) = 1,$$

and discuss the global existence of the solution, and compare with general results for existence and uniqueness.

2. Let $\frac{dy}{dx} = x^2 + \frac{1}{1+y^2}$, $y(0) = 1$.
(a) Show that a solution of this problem exists for at least $0 \le x \le 1$, with $-1 \le y \le 3$ on this interval.
(b) Let

$$z(x_{k+1}) = z(x_k) + h\left(x_k^2 + \frac{1}{1+z(x_k)^2}\right),\ x_k = kh, k = 0, 1, \ldots,\ z(x_0) = 1.$$

Set $E_k = |z(x_k) - y(x_k)|$ where $y(x)$ is the exact solution in (a). Find the smallest constants L, R that you can such that, for any step size $h > 0$, we have

$$E_{k+1} \le (1+hL)E_k + Rh^2$$

on the rectangle $0 \le x \le 1, -1 \le y \le 3$. (Hint: Apply the mean value theorem to $f(\xi, y(\xi)) - f(x_n, y(x_n))$ on page 41 of John's notes.)

3. (a) Discuss the stability of the equilibria and limit cycles of

$$\frac{dx}{dt} = -y + xf(r),\ \frac{dy}{dt} = x + yf(r),\ f(r) = \sin r,$$

where $r^2 = x^2 + y^2$.
(b) The periodic solution $x = \cos(t+c), y = \sin(t+c)$, where c is a constant, of the system

$$\frac{dx}{dt} = -y + xf(r),\ \frac{dy}{dt} = x + yf(r), f(r) = (1-r^2)^2.$$

is sometimes calle "semi-stable". Explain.

4. The *complex Landau equation* $\frac{dz}{dt} = az - b|z|^2 z$ arises in nonlinear stability theory. Here $z(t)$ is complex-valued and a, b are complex numbers. Write the equation as a system of two real equations for $R = r^2(t)$ and $\theta(t)$ where $z = r(t)e^{i\theta(t)}$. Discuss the existence of periodic solutions of this system, in terms of the constants a, b, given that $\Re(a) > 0$.

5. A bead is free to slide without friction on a circular wire hoop. The hoop spins about it's vertical diameter with angular velocity ω, see the figure. The equation governing the position $\theta(t)$ on the hoop is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta - \omega^2 \sin\theta\cos\theta = 0.$$

(a) Discuss the behavior of this system, as ω increases from zero, from the point of view of bifurcation theory.
(b) Write down an energy integral for the system. Find the smallest constant $V > 0$ (in terms of the parameters) such that, if initially $\theta = \pi/2, |d\theta/dt| > V$, then the bead will continually encircle the hoop in one direction.