Ordinary Differential EquationsReview Problems (Cont.)Childress, Spring 2002
Due on or before May 2 in 717 WWH.

6. For the following system carry out the calculation of the Hopf bifurcation by indentifying A_0, A_1, L, R (using the notation of the class discussion), solving for $\mathbf{x}_i(t), \subset= 1, 2, 3$, and ω_2, μ_2:

$$\frac{dx_1}{dt} = 2\mu x_1 + x_2 + 2x_1^2, \quad \frac{dx_2}{dt} = -x_1 - 2x_1^2.$$

In problems 7 and 8 you may apply the following Gronwall inequality, but first prove it: If a is a constant , $g(t) \geq 0$, $h(t)$ continuous, and $h(t) \leq a + \int_0^t g(s)h(s)ds, 0 \leq t \leq T$, then $h(t) \leq ae^{\int_0^t g(s)ds}$.

7. Let $A(t)$ be a continuous $n \times n$ matrix such that $\int_0^\infty \parallel A \parallel (t)dt < \infty$. Prove that any solution of $\frac{dy}{dt} = A(t)y$ which is nonzero tends to a limit $\neq 0$ as $t \to \infty$. Prove also that given any constant vector c, there is a solution z of $\frac{dy}{dt} = A(t)y$ when tends to c as $t \to \infty$.

8. Prove the following theorem on asymptotic stability: Let $\frac{dy}{dt} = Ay + B(t)$, have a solution $z(t)$, Here A is a constant $n \times n$ matrix, all of whose eigenvalues have negative real parts, and B is a continuous $n \times n$ matrix function of t, bounded in the sense that $\int_0^t \parallel B \parallel (s)ds \leq B_0$, B_0 a positive constant, for $t \geq 0$. Then $\lim_{t\to\infty} z(t) = 0$. (You may assume $\parallel B \parallel = \max_i \sum_{j=1}^n |B_{ij}|$.)

9.(a) A real nonlinear second-order scalar equation of the form $F(y_{xx}, y_x, y, x) = 0$ is said to be homogeneous in x if the equation is invariant under the transformation $x \to Ax$ where A is any real number. (Here $y_{xx} = \frac{d^2y}{dx^2}$ etc.)Show that any homogeneous equation of this form can be rduced to an autonomous equation of the form $G(y_{tt}, y_t, y) = 0$, by the substitution $x = e^t$.

(b) Show that an autonomous equation of the second order of the form $G(y_{tt}, y_t, y) = 0$ can always be reduced to a non-autonomous equation of first order of the form $H(u_y, u, y) = 0$ by expressing $u = y_t$ as a function of y and finding an equation in $u(y)$.

(c) Apply the above methods to discuss the solutions of

$$y_{xx} + y_x(y+1)/x + y/x^2 = 0.$$

10. Apply the method of averaging to obtain an approximate solution of

$$\frac{d^2x}{dt^2} + x + \epsilon x^3 = 0, \ x(0) = r_0 \cos\theta, \frac{dx}{dt}(0) = -r_0 \sin\theta_0$$

in the form

$$x(t) = r_0 \cos(t + \frac{3}{8}\epsilon r_0^2 t + \theta_0) + O(\epsilon),$$

$$\frac{dx}{dt}(t) = -r_0 \sin(t + \frac{3}{8}\epsilon r_0^2 t + \theta_0) + O(\epsilon).$$