Ordinary Differential Equations **Homework 2** **Childress, Spring 2002**

Due February 5

1. The simple pendulum equation as given in class is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0. \tag{1}$$

Verify the phase-plane integral given in class by multiplying (1) by $d\theta/dt$ and integrating, to obtain

$$\frac{1}{2}\Phi^2 + (1-\cos\theta) = E \tag{2}$$

where the "total energy" E is a positive constant, and $\Phi = \sqrt{L/g}(d\theta/dt)$. For $E < 2$ show that the curves defined by (2) are closed, and, using the symmetry of the orbit, that the period of the orbit once around the curve is given by

$$T = \sqrt{8L/g}\int_0^{\theta_E} \frac{d\theta}{\sqrt{E-1+\cos\theta}},$$

where $\theta_E = cos^{-1}(1-E), \theta_E > 0$. Verify that for small oscillations about $\theta = 0$, where $\cos\theta \approx 1-\theta^2/2$, the period is approximately $\frac{2\pi}{\omega}, \omega = \sqrt{g/L}$. Show that as $E \to 2$ the period of the pendulum becomes arbitrarily large. (Hint: $T \geq \sqrt{8L/g}\int_0^{\theta_E}\frac{d\theta}{\sqrt{1+\cos\theta}}$.)

2. Show that a non-autonomous first-order system of n ODE's, for

$$\mathbf{y} = (y_1(x), \ldots, y_n(x))$$

say, is equivalent to an autonomous first-order system on $n+1$ ODE's, by introducing $y_{n+1} = x$.

3. Smoothness of $f(x,y)$ does not insure that Picard iterates converge. Consider the IVP

$$\frac{dy}{dx} = 1 + (y-x)^2, \quad y(0) = 0.$$

(a) In the rectangle $R : |x| \leq a, |y| \leq b$, identify a suitable Lipshitz constant L in terms of a, b and find (using elementary estimates) the largest α that you can which insures the Picard existence and uniqueness on I_α. (b) Observe that the unique solution is $y = x$. Show that the Picard iterates can be obtained from $\eta_n(x) = \int_0^x \eta_{n-1}^2(\xi)d\xi$, $\eta_n = y_n - x$, $\eta_0 = -x$. By studying this recursion, show that the $y_n(x)$ converges to the above solution if $|x| < r_0$ but diverges when $|x| > r_0$ where $r_0 = \prod_{n=2}^{\infty}(2^n-1)^{2^{-n}} \approx 2.5748$.

4. Let $\mathbf{M}$ be and an $n \times n$ real matrix and define

$$|\mathbf{M}| = \sup_{|\mathbf{y}|} \frac{|\mathbf{M}\cdot\ \mathbf{y}|}{|\mathbf{y}|}$$

in terms of real n-vectors $\mathbf{y}$. (a) Show that for $|\mathbf{y}| = \max_i |y_i|$, then $|\mathbf{M}| = \max_i \sum_{j=1}^{n} |M_{ij}|$. (b) Show that this also defines a norm on the space of $n \times n$ matrices.

5. Consider the first-order Ricatti equation $dy/dx = x + y^2$, with $y(0) = 0$. (a) Find, again using only elementary estimates, suitable L and largest α for Picard existence and uniqueness, in terms of a, b defining R. (b) Let $y = -u'(x)/u$, and show that $d^2u/dx^2 + xu = 0$. This is the Airy's differential equation. Regarding it as a vector first-order equation, $d\mathbf{y}/dx = f(x, \mathbf{y})$, with $|\mathbf{y}| = |y_1| + |y_2|$, and add the initial conditions $u(0) = 1, u'(0) = 0$.. Again find, for given a, b, suitable L and largest α for Picard existence and uniqueness of the solution to the second-order system.