Ordinary Differential Equations **Homework 3** **Childress, Spring 2002**

Due February 12

1. Prove that all solutions of the first-order system of two equations

$$\frac{dy_1}{dx} = -y_2 + y_1(1-A), \quad \frac{dy_2}{dx} = y_1 + y_2(1-A),$$

where $A = y_1^2 + y_2^2$, exist for all x.(Hint: Let $|y| = A^{1/2}$ and consider $\frac{dA}{dx}$. Using problem 1 of Hwk 1, show that A and hence $|y|$ is bounded for all x.)

2. Sketch the phase plane (the plane of $(y_1 = y, y_2 = y_x)$)of the second-order equation

$$\frac{d^2y}{dx^2} + y^2 = 0.$$

Describe the behavior of solutions from the point of view of global existence. (For example, the curves $y_2 = \pm\sqrt{2/3}(-y_1)^{3/2}$, $y_1 \leq 0$ form a cusp at the origin of the phase plane and divide it into two parts. A solution $(y_1(x), y_2(x)$ on the upper branch exists for all time and tends to $(0,0)$. A solution starting at $(y_1 \neq 0, y_2$ on the lower branch exists for a finite time only. Use the rolling ball analogy to understand the behavior in the two parts.)

3. The rotation of a rigid body in three dimensions is governed by the first-order equation $\frac{d\mathbf{M}}{dt} + \omega \times \mathbf{M} = 0$ where $\times$ denotes the vector cross product. Here $\mathbf{M}(t) = (M_1, M_2, M_3)$ is the angular momentum vector relative to the spinning body, and $\omega(t) = (\omega_1, \omega_2, \omega_3)$ is the instantaneous angular velocity of the body. (This equation expresses the fact that relative to inertial space the angular momentum vector is a constant if no torques are applied to the body.)

Relative to principle axes of the body we have $M_i = I_i\omega_i, i = 1, 2, 3$, where I_i are the angular moments of inertia about the three axes. We assume the I_i have three different values.

(a) Show that the system has the component form

$$\frac{dM_1}{dt} + M_2M_3(\frac{1}{I_2} - \frac{1}{I_3}) = 0, \quad \frac{dM_2}{dt} + M_3M_1(\frac{1}{I_3} - \frac{1}{I_1}) = 0, \quad \frac{dM_3}{dt} + M_1M_2(\frac{1}{I_1} - \frac{1}{I_2}) = 0.$$

(b) Show that $J_1 = M_1^2 + M_2^2 + M_3^2$ and $J_2 = \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}$ are constant for any solution.

(c) Show that solutions exist for all time.

(d) Examine by linearization the behavior of solutions near the three constant solutions where one and only one component of $\mathbf{M}$ is non-zero. For example, near $\mathbf{M} = (M_1^{(0)}, 0, 0)$, M_1 will approximately stay constant while M_2, M_3 will satisfy in the linear approximation

$$\frac{dM_2}{dt} + M_3M_1^{(0)}(\frac{1}{I_3} - \frac{1}{I_1}) = 0, \quad \frac{dM_3}{dt} + M_1^{(0)}M_2(\frac{1}{I_1} - \frac{1}{I_2}) = 0.$$

Thus show: the solution $M_i^{(0)}$ is linearly (neutrally) stable if $I_i = \max(I_1, I_2, I_3)$ or $I_i = \min(I_1, I_2, I_3)$, but is unstable if I_i has the intermediate value.

(e) Let $I_1 = 1, I_2 = \frac{1}{2}, I_3 = \frac{1}{3}$. Show that then the two planes $M_1 \pm M_3 = 0$ are invariant under this system, i.e. a solution starting from a point on one of these planes, stays on that plane.

(f) Sketch the solution trajectories on the sphere $M_1^2 + M_2^2 + M_3^2 = 1$, for the special case (e).

(g) Do an experiment to verify (d) using a book which is taped closed. For most books I_{max} corresponds to the direction perpendicular to the face of the book, I_{min} points across the smallest side of the face. Trying to spin the book about the longer direction (the intermediate case) results in *Eulerian wobble.*

4. Apply the Euler polygon construction to $\frac{d^2y}{dx^2} + y = 0$, $y(0) = 0, y'(0) = 1$, with $0 \leq x \leq 1$. Using $|\mathbf{y}| = max(|y|, |y'|)$, find a step size h which makes $|y - z| < 10^{-3}$ throughout this interval, z being the approximate solution.