Ordinary Differential Equations Homework 4 Childress, Spring 2002
Due February 19

1. Locate and classify all of the critical points of

$$\dot{x} = y,\ \dot{y} = -ay - b\sin x$$

as a function of the constants a, b, subject to the conditions $a \geq 0, b > 0$. If $a > 0$, this system describes a simple nonlinear pendulum with damping (with $x = \theta, y = \dot{\theta}$). Based on the intuition provided by this fact, if $a > 0$, what happens to orbits passing through points on the positive y axis, as $t \to \infty$?

2. The case $\dot{x} = Ax + By, \dot{y} = Cx + Dy$ with $AD = BC$ is singular or *degenerate* . Investigate this case by finding equations for $Dx - By$ and $Ax + By$. Sketch orbits in in the phase plane for the case $A = 1, B = 2, C = 3, D = 6$.

3. Consider the system $\dot{x} = -x - y^2,\ \dot{y} = x$. Classify the critical point at the origin and sketch orbits of the linearized system in the phase plane. Then assess the effect of the nonlinear term and sketch the phase plane of the full system, by sketching in direction vectors or using any graphing aids you may have.

4. Prove that the second-order equation

$$\frac{d^2z}{dt^2} + (z^2 + 2(\frac{dz}{dt})^2 - 1)\frac{dz}{dt} + z = 0$$

has a nontrivial periodic solution. First convert to a first-order system for $x(t) = z, y(t) = \frac{dz}{dt}$.(Hint: consider a suitably chosen annulus $r_1^2 < x^2 + y^2 < r_2^2$.)

5. Green's theorem in the plane states that, if $f(x, y), g(x, y)$ are continuous and have continuous derivatives in a simple region R bounded by a simple closed curve C, then $\oint_C [f(x, y)dy - g(x, y)]dx = \int\int_R (f_x + g_y)dxdy$. Suppose $\dot{x} = f(x, y), \dot{y} = g(x, y)$, and that C is a periodic orbit of this system.

(a) Show that in this case the line integral above will vanish, and $f_x + g_y$ cannot have one sign over R.

(b) Prove that the system $\dot{x} = x - xy^2 + y^3, \dot{y} = 3y - yx^2 + x^3$ has no periodic solution within the circle $x^2 + y^2 = 4$.

(c) Show that this last result cannot be proved by examining the sign of $d(x^2 + y^2)/dt$ on $x^2 + y^2 = 4$.