Ordinary Differential Equations **Homework 5** **Childress, Spring 2002**

Due February 26

1. The 2D motion of a cannonball in a gravitational field (0,-g), with the force of air resistance taken to be proportional to the velocity, can be described by the equations

$$\frac{d^2X}{dt^2} = -\lambda\frac{dX}{dt}, \quad \frac{d^2Y}{dt^2} = -g - \lambda\frac{dY}{dt}.$$

Here $(X(t), Y(t))$ is the position of the cannonball in the (X, Y) space, λ is the positive drag coefficient, and g the constant acceleration of gravity. Assuming λ is small, use perturbation theory to compute the order λ corrections to range and time-of-flight (here these are the corrections proportional to λ) when the initial conditions are

$$X(0) = 0, \frac{dX}{dt}(0) = V\cos\theta, \quad Y(0) = 0, \frac{dY}{dt}(0) = V\sin\theta$$

Here $V > 0$ is the initial speed and θ is the initial inclination.

In the general notation $\frac{dy}{dx} = f(y, \lambda)$, y is here a 4-vector and the phase space is four dimensional. However it is easiest to work with X, Y directly and two second-order equations, using expansions of the form $X(t, \lambda) = X_0(t) + \lambda X_1(t) + o(\lambda)$, and $Y(t, \lambda) = Y_0(t) + \lambda Y_1(t) + o(\lambda)$. Insert these expressions into the equations and collect the terms independent of λ and those proportional to λ to obtain four differential equations.

The range is the value of X when the cannonball hits the ground $(Y = 0, t > 0)$, and the time of flight is the time at which this occurs.

2. Give another proof that volumes V in phase space are conserved by an autonomous system having $div\ \mathbf{f} = 0.$, based on the following idea: If we can show $\frac{dV}{dx} = 0$ at an arbitrary initial x, say $x = 0$, then we are done. Let a phase space volume be equal to V_0 at the initial point $x = 0$, where $\mathbf{y} = \eta$. For $x > 0$ we then have the change of varaiable formula

$$V(x) = \int_{V_0} Det(\mathbf{J}) d\eta_1 d\eta_2 \ldots d\eta_n, \quad J_{ij} = \frac{\partial y_i}{\partial \eta_j}.$$

Assuming $\mathbf{f}$ and $\mathbf{f}_y$ continuous, use the fact that

$$\frac{\partial y_i}{\partial \eta_j} = \delta_{ij} + \frac{\partial f_i}{\partial y_j}(\eta)\, x + O(x^2),$$

where $\delta_{ij} = 1$ if $i = j$, $= 0$ otherwise.

3. A two-dimensional *gradient system* has the form

$$\frac{dx}{dt} = U_x, \quad \frac{dy}{dt} = U_y, \quad x(0) = \xi, y(0) = \eta$$

where $U(x, y)$is a given function with continuous second derivatives.

(a) Show that this system can be written as a Hamiltonian system if U is harmonic, i.e. $U_{xx} + U_{yy} = 0$.

(b) Verify by a direct calculation of the derivative on the left that

$$\frac{dJ}{dt} = (U_{xx} + U_{yy})J, \quad J = x_\xi y_\eta - x_\eta y_\xi.$$

(c) Prove that if some finite point (x^*, y^*) belongs to the limit set of an orbit of this system, then (x^*, y^*) is an equilibrium (rest) point of the system, and so an extremum of U, but never a local minimum. (Note: $\frac{dU}{dt} = U_x^2 + U_y^2$.)

4. Determine the behavior in time of the any volume of any region of the three-dimensional phase space, under the orbits of the Lorenz system

$$\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = rx - y - xz, \quad \frac{dz}{dt} = -bz + xy,$$

σ, r, b being positive constants. What is the implication for the Lorenz attractor?