Ordinary Differential Equations **Homework 6** **Childress, Spring 2002**

Due March 5

1. For the double pendulum system presented in class, the positions of the two bobs are

$$(x_1, y_1) = (L_1 \sin\theta_), -L_1\cos\theta_1), \quad (x_2, y_2) = (L_1 sin\theta_1 + L_2\sin\theta_2, -L_1 cos\theta_1 - L_2\cos\theta_2).$$

(a) Given that the pendulum is in the constant gravitational field $(0, -g)$, and using $q_i = \theta_i$ and the p_i derived in class from the Lagrangian, show that the Hamiltonian H for the system is

$$H(q_1, p_1, q_2, p_2) = \frac{m_2 L_2^2 p_1^2 - 2m_2 L_1 L_2 \cos(q_1 - q_2) p_1 p_2 + (m_1 + m_2) L_1^2 p_2^2}{2L_1^2 L_2^2 (m_1 + m_2\sin^2(q_1 - q_2))} + V,$$

$$V = -m_1 g L_1 \cos q_1 - m_2 g (L_1 cos q_1 + L_2\cos q_2).$$

(b) Give the approximate Hamiltonian appropriate to small oscillations about $q_i = p_i = 0$. Do this by considering q_i, p_i small and retaining at most terms quadratic in these quantities in H.

2. (a) From the approximate Hamiltonian in problem 1(b) show that the Hamilton's equations for the linearized system reduce to

$$\ddot{q}_1 \approx \frac{g}{L_1(\mu - 1)}(q_2 - \mu q_1), \quad \ddot{q}_2 \approx \frac{g\mu}{L_2(\mu - 1)}(q_1 - q_2),$$

where $\mu = 1 + \frac{m_1}{m_2}$.

(b) From the linear system in (a), show that, if $L_1 = L_2$, the system is stable at the rest point $q_i = p_i = 0$. (Hint: Set $q_i(t) = Q_i e^{\lambda t}$, evaluate the determinant of a 2×2 matrix, and show that all roots λ are pure imaginary. Note that two periods are involved.)

(c) Show that in fact linear stability obtains for arbitrary L_1, L_2.

3. Consider the 2nd-order linear equation $u'' + p(x)u' + q(x)u = 0$ for $u(x)$, where $p(x), q(x)$ are continuous for all x.

(a) Write the equation as a first order system for $\mathbf{y} = (y_1, y_2) = (u, u')$.

(b) Show that the equation satisfied by the Wronskian is identical to the equation satisfied by the determine of the Jacobian $\frac{\partial y_i}{\partial \eta_j}$ of the solution satisfying $\mathbf{y}(0) = \eta$.

(c) Show that if the orbits of solutions determine an area-preserving flow, then the Wronskian of any two solutions is a constant.

4. In each case, find a second solution using variation of parameters:

$$\text{(a)} \quad y'' + 2y' + y = 0, \quad y_1(x) = e^{-x}.$$

$$\text{(b)} \quad y'' + 2xy' + 2y = 0, \quad y_1(x) = e^{-x^2},$$

5. Show that, given one solution $y_1(x)$ of the equation, the method of variation of parameters can by used to reduce a third-order equation $y''' + p(x)y'' + q(x)y' + r(x)y = 0$ to a second order equation for some undetermined function, whose solution yields a solution of the third-order problem.