Ordinary Differential Equations **Homework 7** **Childress, Spring 2002**

Due March 26

1. (Ex. 4.2.1 of John's notes) Find a solution of the Bessel equation of order 0 :

$$z'' + \frac{1}{x}z' + z = 0,$$

satisfying $z = 1,\ z' = 0$ for $x = 0$, by assuming a power series representation. Use the functional relation obtained by the Wronskian to find another (linearly independent) solution of the equation. Show that this solution becomes infinite as $x \to 0$ like $\log x$.

2.(Ex. 4.2.2) Observe that the differential equation

$$L[y] = y''' - \frac{2}{x}y'' + y' - \frac{2}{x}y = 0$$

has the two solutions $\cos x,\ \sin x$.

(a) Find the general solution by rewriting the equation as an equation for $y'' + y$.

(b) Find the general solution using instead by writing the scalar equation as a system of three first-order equations and using the formal procedure from class.

(c) Solve the inhomogeneous equation $L[y] = x^2$ in any way.

3.(Ex. 4.2.3) Show that if $w(x)$ is a vector such that $w^T(x)y(x) =$ constant for every solution $y(x)$ of $y' = Ay$, then $w(x)$ is a solution of the adjoint system $w' = -A^T w$.

4. (Method of integral transforms) Consider an integral with respect to the complex variable s of the following form:

$$w = \int_C G(s)e^{sz+F(s)}ds,$$

where C is a piecewise smooth curve in the s-plane which begins and ends at ∞, by tending asymptotically to two straight lines R_1, R_2 extending from the origin to ∞.

We seek a solution of *Airy's* equation $Lw \equiv w'' - zw = 0$, having the above form.

By differentiating under the integral sign, show that the functions $G, F = 1, -\frac{1}{3}s^3$ yields a perfect differential, i.e.

$$Lw = \int_C G(s)Le^{sz+F(s)}ds = \int_C dH(s,z),$$

and determine choices of R_1, R_2 such that

$$\lim_{s\to\infty} H(s,z) = 0$$

on R_1, R_2. Show that one solution of Airy's equation is given by the real integral

$$A_i(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + tz\right)dt.$$

(Hint: Choose R_i so that $ds = (i - \epsilon)dt$ on R_1, extending downward, and $ds = (i + \epsilon)dt$ on R_2, extending upward, where t is a real variable. Then take the limit $\epsilon \to 0$.)

5. Ex. 5.1.1. (Hint: The existence of a C, such that $C^{-1}AC$ has Jordan Normal Form Λ, is assumed. Find a D such that $D^{-1}B_kD = \tilde{B}_k$, where $\tilde{B}_k$ is a block in $\tilde{\Lambda}$, as described in the problem.)

6. Ex. 5.1.2.