Ordinary Differential Equations Homework 8 Childress, Spring 2002

Due April 2

1. The matrix A and its Jordan normal form are given by

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, C^{-1}AC = \Lambda = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Following the notes from class, solve the equation $dy/dx = Ay$ in several ways as follows:

(a) Consider the relation $AC = C\Lambda$ and solve $dy/dx = Ay$ to obtain the fundamental solution matrix

$$Y = e^{2x} \begin{pmatrix} 1 & x & x^2/2+2 \\ 0 & 1 & x+1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Let $y = Cz$ in $dy/dx = Ay$ and solve the system for z in steps. Show that the constants of integration may be chosen to yield the Y in (a).

(c) Compute $Y^* = e^{Ax}$ from its power series and check that $Y^*C = Y$;

(d) Use $Y^* = e^{Ax} = \frac{1}{2\pi i}\oint_\Gamma (\zeta I - A)^{-1} e^{\zeta x} d\zeta$ and compute residues. (Recall: the residue is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion at z_0.)

2. Three identical bobs of mass m lie on a frictionless surface and are attached to identical springs and walls as shown in the figure. The masses move on the $x-$axis, and have positions $x_i(t), i = 1, 2, 3$. The springs obey Hooke's law with constant k, that is the force exerted is proportional to the distance the spring is stretched from its rest position.

(a) Write the equations of motion in the vector form of a second-order system, i.e.

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{k}{m}A\mathbf{x}, \quad \mathbf{x} = (x_1, x_2, x_3)^T.$$

Here A is a 3×3 matrix of real constants.

(b) By finding the eigenvalues and eigenvectors of A, determine three modes of oscillation of the system and describe them in terms of the movement of the bobs.

3. Show that in general the $n \times n$ linear system $dy/dt = A(t)y$ is *not* solved by $y = e^{B(t)}$, where $B(t) = \int_0^t A(s)ds$, the exponential of a matrix being as usual defined by its power series. (Recall: in general matrices do not commute.)

4. Consider *Mathieu's equation*: $d^2z/dt^2 + (\Omega^2 + \epsilon \cos t)z = 0$, where Ω, ϵ are real constants. Show that if $\Omega \neq k/2$, k an integer, then all solutions are bounded for ϵ sufficiently small. (Hint: consider the equation for $\epsilon = 0$ and use continuous dependence of solutions upon parameters.)

5. Ex. 6.1.3.