Ordinary Differential Equations Homework 9 Childress, Spring 2002

Due April 9

1. Compute as a power series in ϵ through terms of order ϵ^2 a fundamental solution matrix Z of the Mathieu equation

$$\frac{d^2z}{dt^2} + (1 + \gamma(\epsilon) + \epsilon \cos 2t)z = 0,$$

satisfying $Z(0) = I,\ Z = [z^{(1)} \quad z^{(2)}]$ where $z^{(2)}$ is stipulated to be periodic with period π. Compute the Floquet matrix $\Omega(\epsilon)$ through terms of order ϵ^2. Be sure to allow $\gamma = \gamma_1\epsilon + \gamma_2\epsilon^2 + \ldots$. Also find a matrix M such that $M^{-1}\Omega M$ is in Jordan Normal Form.

2. (a) Find explicitly the solution matrix $\Phi(t)$ satisfying $\Phi(0) = I$ for the system

$$\frac{dx}{dt} = -3x,\ \frac{dy}{dt} = -y + 2z,\ \frac{dz}{dt} = -2y - z.$$

(b) Show that the origin is asymptotically stable.

(c) Prove that the origin is asymptotically stable by Liapunov's direct method. Note that the coefficient matrix A of the system is already in real Jordan form.

3. Consider *Liénard's equation*

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0,$$

where f, g are continuous functions, $g(0) = 0,\ f(0) > 0$, and $xg(x) > 0$ in a neighborhood of the origin excluding $x = 0$.

(a) Show that the system is equivalent to

$$\frac{dx}{dt} = \eta(t) - F(x),\ \frac{d\eta}{dt} = -g(x),$$

where $F(x) = \int_0^x f(s)ds$, and that there is an equilibrium point at the origin of the (x, y)-plane.

(b) Taking $Q = Q_1 = \frac{1}{2}\eta^2 + G(x)$ as a Liapnuov function, where $G(x) = \int_0^x g(s)ds$, show that the origin is (locally) stable, according to Liapunov's Theorem.

(c) Show that , with $y = \frac{dx}{dt}$ and $Q = Q_2 = \frac{1}{2}y^2 + G(x)$, we may again establish stability of $(0, 0)$ by Liapunov's theorem.

(d) Find a Liapunov function $Q_3(x, y)$ with which we may establish local *asymptotic* stability of $(0, 0)$.

4. Show that the origin is an asymptotically stable point of equilibrium of the nonlinear system

$$\frac{dx}{dt} = y - x^3,\ \frac{dy}{dt} = -x^3,$$

but that it is an unstable point of equilibrium of the linearized system there. (Hint: Consider Liapunov functions of the form $Q = x^m + cy^n$.)