Constraints on stretching by paired vortex structures II. The asymptotic dynamics of blow-up in three dimensions

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Abstract

This paper continues our study of the use of paired vortex structures in the construction of incompressible Euler flows in three dimensions which produce substantial vorticity growth. These flows, which have been considered before as candidates for Euler blow-up, are here derived in a geometry which is the product of moving planar curve $C(t)$, the center vortex line, and locally almost two-dimensional fluid motion in planes $P(t)$ orthogonal to the curve. In the present part we show that the construction of a non-self-similar flow of the kind proposed here can be reduced by contour averaging to a generalized differential system. Each section consists of an invariant 2D Euler flow, assumed here to have the contour structure of paired vortices of opposite sign, of a size compatible with the requisite propagation speed of the singular cocoon. Axial flow within the cocoon alters the local contours from section to section. This leads to a complex interaction of the generalized differential system with the kinematics of the center vortex.

To simplify this interaction we propose here that the Batchelor couple introduced in part I is one of a family of invariant 2D flows suitable for the construction of a singular Euler flow in 3D. This family includes vortices of arbitrarily small cross-section, allowing the self-induced stretching of the vortex pair to be decoupled from the variation of the circulation profiles by the axial flows. We thus propose to show in the present paper that the thin-vortex limit of these extensions of the Lamb-Chaplygin-Batchelor couple (acknowledging the prior presentations of this solution of the steady Euler equations) offers the first clean example of blow-up of Euler flows in three dimensions.

A crucial issue is the axial flow in the vortices, which we shall treat in this part using a one-dimensional model. This has the advantage of leading, for the fixed-point problem at the core of the existence of the blow-up, to an ODE problem which is readily solved numerically. Surprisingly, our model shows that the squeezing down of the cross-section of the self-stretching vortices overcomes the axial flow induced by the lower pressures of the most stretched section, so that the axial flow is in fact away from the singularity region.

Some implications of such solutions for Euler flow theory are discussed. We conjecture that the solutions are highly unstable and not directly observable in numerical simulations. We also give reasons for the absence of any such singularities for the Navier-Stokes equations. The final phase of this research will deal with the full PDE problems for the non-self-similar Euler flow associated with the blow-up.

1 Introduction

The present paper continues our study of vortex stretching in flows of simple topology. In [Childress (2006)], hereafter referred to as I, we argued that Euler blow-up, if it occurs at all, is in some sense extremely rare. In the present paper we explore a family of flows which are related to paired vortex flows previously proposed as candidates for finite time singularities. Studies of this kind are important even if they lead to negative results, since it is essential to understand the often invoked "depletion of nonlinearity" as an explanation of the lack of any convincing evidence of Euler blow-up. However the present work suggests that in fact there do exist Euler flows (probably unstable) which produce infinite vorticity in finite time. The flows we study involve non-self-similar vortex stretching of paired filaments without the constraint of non-swirling axial symmetry, obtained in the kinematic model put forward in section 3 of I. We use the term "kinematic" in a slightly perverse way, referring to the blow-up of paired vortices as obtained in I, without a dynamical basis for the relation between the advection of vortices by the normal, and the local Jacobian of the center vortex of the pair.

We first review some of the ideas presented in I. According to [Beale, Kato & Majda (1984)] blowup is accompanied by infinite vorticity, and this is achieved by the stretching of vortex lines. The stretching can be achieved by the variation of the velocity component tangent to the line (shear stretching), or else by advection in the direction of the negative normal of a curved vortex line (expansive stretching). Shear stretching is important in the model studied by [Pelz (2001)] and may well dominate the stretching events in turbulent flows on many scales. But it involves sheared vortical structure in order to make the velocity induced by vortex tube A stretch tube B and *vice versa*. Expansive stretching was invoked and pioneered in the models of [Siggia (1985), Pumir & Siggia (1990)], and of [Kerr (2005)], and has the attraction of being accessible by vortex lines which are locally parallel, provided that paired structures of opposite sign are so aligned. In particular nearly two-dimensional flow structures are capable of self-stretching. This makes paired vortex structures attractive for analysis of singular behavior in Euler flows. Numerically, however, singularity formation seems to be stalled by deformation of the structures in the final phase. The structures studied by Kerr are not locally 2D, but are similar in some respects to the "hairpin" singularities of the kinematic models considered in I. We will discuss below the relation of our work to that of [Pumir & Siggia (1990)].

To investigate how fast paired structures can self-stretch, we considered in I the simplest family of Euler flows in 3D where this happens, namely axisymmetric flow without swirl, where the vorticity has the form $(0, 0, \omega_\theta)$ in cylindrical polar coordinates and all vortex lines are circles about a common axis. It is well known ([Majda & Bertozzi (2002)]) that ω_θ then grows at most like an exponential in time, so blow-up does not occur. In I we examined the implications of constant vorticity support volume on the maximal rate of growth. We estimated kinematically the optimal configuration for maximal expansive stretching of a target ring given an initial bound on $|\omega_\theta/r|$, and found that vorticity grew in fact no faster than $O(t^2)$. We called this optimizing arrangement of vorticity a *kinematic cocoon*. The kinematic cocoon conserves volume, but not kinetic energy, which grows in proportion to the radius of the target ring.

We also found in I that there is another cocoon construction which conserves energy but not volume, leading to growth as $O(t^{4/3})$. It appears that both constraints can be satisfied kinematically by cocoons which shed filamentary vorticity, although the constraint of energy conservation is essential only when axial symmetry is imposed. This point will be important in the present paper.

The $O(t^2)$ maximal growth of the kinematic cocoon of constant volume suggests comparison with quasi-2D flows. A classical solution of Euler's equations in two dimensions, consisting of a pair of oppositely sign vortical regions contained within a circular boundary $r = a$, may be found in [Batchelor (1967)], a solution that goes back to the work of Lamb and Chaplygin. [1] The dipolar structure moves without change of form at a constant velocity. In three dimensions we may consider an analogous thin toroidal structure as an initial vorticity of an axisymmetric flow without swirl, which then expands while maintaining a self-similar structure. Since vortical flux is conserved, $\omega_\theta a^2 = O(1)$. The speed of propagation $\sim \omega_\theta a$, and the torus volume must be conserved, $Ra^2 = O(1)$ where R is the large radius of the torus. It follows the speed at large R is $\sim \sqrt{R}$, leading to $R = O(t^2)$, at a rate satisfying our bound. This estimate omits the fact already noted, that the kinetic energy of the toroidal structure, which may estimated as $O(\omega_0^2 a^4 R) \sim O(R)$ increases, so the circular boundary of the vortex does not survive–there must be core deformation to conserve energy, along with a lessening of vortex stretching.

If one relaxes the constraint of axial symmetry and assumes that paired structure moves as a Lamb-Chaplygin-Batchelor (LCB) couple in all local sections, then, as we showed in I, of a line moving in this way by the normal produces a finite time singularity. At the singular time vorticity is infinite at a point, but only a finite amount of total stretching of vortex lines has occurred. The question raised in the present part is essentially, can this kinematic picture survive if the paired structure satisfies Euler's equations in an appropriate sense associated with the asymptotics of the singularity? The dynamics involves possible core deformation as well as the creation of axial flows induced by the pressure gradients developed when a vortex is stretched locally. We shall show

[1] In I we termed this structure a *Batchelor couple*, because it was a prominent example in [Batchelor (1967)]. As might be expected, the solution has a richer history, which was discussed in [Meleshko & van Heijst (1994)]. Horace Lamb and S.A. Chaplygin describe this structure and its variants over a century ago, see [Lamb (1906), Chaplygin (1903)]

that if the paired structure is initially a LCB couple at each section, it will subsequently evolve according to a generalized differential system. The dynamics must, under this system, depart from the kinematics described in I.

Nevertheless, the dynamical description developed here contains the possibility of using other 2D Euler solutions as the underlying cross-sectional flow, and we argue here that there exists a family of flows including the Lamb-Chaplygin-Batchelor flow. Included are paired vortex tubes of near circular vortex section, which allow the implications of the generalized system to be studied analytically. This is a key step which breaks the deadlock over coupling of the velocity of the vortex pair with the core dynamics. The two essential features are, first, the fact that it is only the total circulations of the vortices, the constants $\pm\Gamma$ say, which determine the velocity of the pair, and second, for small cores the core boundary is essentially a circle, allowing an axisymmetric treatment in classical terms,

2 The dynamic cocoon

We now refer to curve $C(t)$, studied in section 3 of I, with $\gamma \in (1/2, 1)$, as the *center vortex*. It will actually be a vortex line on which vorticity vanishes. We introduce the time-dependent orthogonal curvilinear coordinate system derived from the center vortex, with triad $(\mathbf{n}, \mathbf{b}, \mathbf{t})$, coordinates (ξ, η, ζ), and metric $ds^2 = d\xi^2 + d\eta^2 + h^2 d\zeta^2$, where $h = 1 - \xi\kappa$. The cocoon boundary will be the surface $\mathcal{C} : \xi^2 + \eta^2 = \frac{1}{4}\kappa^{-2}, -\infty < \zeta < +\infty$, the numerical factor insuring that h remains positive within the cocoon. We wish to solve Euler's equations within the cocoon starting at an initial time $t = T < 0$, and to do so we need to supply an initial vorticity field, then track its evolution under the constraint of Euler's equations and some boundary conditions associated with the cocoon boundary. We assume that the fluid density is unity. As we shall make clear presently, the initial vorticity will be confined within another surface $\mathcal{B} : \xi^2 + \eta^2 = r_B^2(\zeta), -\infty < \zeta < +\infty$, and we may in fact choose $\max_{-\infty<\zeta<+\infty} \kappa r_B$ to be as small as we like. Indeed on the curve g is positive and has the estimate (**??**) for sigma large, from which it follows that κ/g is bounded as a function of σ. Thus we may, in the LCB couple, fix $\tilde{a}$ to match the curve velocity, the decrease it everywhere by a constant multiplier while simultaneously increasing the flux K by the inverse factor.

For $t > T$ the surface $\mathcal{B}$ will be material and track the motion of the center curve C with $\beta = 2$. The tube bounded by $\mathcal{B}$ will thus be stretched and the transverse ξ, η dimensions contract, so that as we can understand its evolution from the dynamics of C. Therefore, the crucial issue is the behavior of the curve in the neighborhood of $\mathcal{B}$ as $\tau \to 0$ with σ fixed. Assuming that we have indeed set up the initial vorticity so as to track the motion of C. Then, our analysis shows that $\xi, \eta \sim O(\tau^\gamma)$ and $\kappa^{-1} \sim \zeta \sim O(\tau^{1-\gamma})$ where $A \sim B$ means that A is neither much small than, nor much larger than B as $\tau \to 0$. We will wee that vorticity kinematics will then imply that $u \sim v \sim O(\tau^{-\gamma})$, and it will also transpire that $w \sim O(\tau^{-\gamma})$ and that pressure satisfies $p \sim O(\tau^{-2\gamma})$. These estimate allow us to estimate the size of all terms in Euler's equations in this coordinate system.

In Cartesian coordinates Euler's equations for an incompressible fluid are (assuming that the fluid density is unity)

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

To write these equations in the new coordinate system we need a few differentiation formulas derived using the chain rule. For any scalar function f we have

$$\frac{\partial f}{\partial t}\Big|_{\xi,\eta,\zeta_0} = \frac{\partial f}{\partial t}\Big|_{x,y,z} + \frac{\partial \mathbf{x}}{\partial t}\Big|_{\xi,\eta,\zeta_0} \cdot \nabla f. \tag{2}$$

We assume that the curve moves according in the direction of the normal as in I, but now take the curve velocity as $u = U(\zeta, t)$ in the direction of the normal , since we want to use u, v, w for fluid velocity in the present coordinate system. Thus $\frac{\partial \mathbf{x}}{\partial t}\Big|_{\xi,\eta,\zeta_0} = U\mathbf{n}$. Then

$$\frac{\partial f}{\partial t}\Big|_{x,y,z} = \frac{\partial f}{\partial t}\Big|_{\xi,\eta,\zeta_0} - U\frac{\partial f}{\partial \xi}. \tag{3}$$

We also have

$$\frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial t}\Big|_{x,y,z} = \frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial t}\Big|_{\xi,\eta,\zeta_0} + U\frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial \xi} = \frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial t}\Big|_{\xi,\eta,\zeta_0}, \tag{4}$$

so that, since

$$\frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial t}\Big|_{\xi,\eta,\zeta_0} = (U_\zeta \mathbf{t}, 0, -U_\zeta \mathbf{n}), \tag{5}$$

we have

$$\frac{\partial(u\mathbf{n} + v\mathbf{b} + w\mathbf{t})}{\partial t}\Big|_{\xi,\eta,\zeta_0} = \mathbf{n}(\mathcal{D}u + wU_\zeta) + \mathbf{b}\mathcal{D}v + \mathbf{t}(\mathcal{D}w - uU_\zeta), \tag{6}$$

where

$$\mathcal{D} = \frac{\partial}{\partial t}\Big|_{\xi,\eta,\zeta_0} - U\frac{\partial}{\partial \xi}. \tag{7}$$

We also have the standard formulas

$$\begin{aligned}\mathbf{u} \cdot \nabla \mathbf{u} = &\Big[uu_\xi + vu_\eta + h^{-1}wu_\zeta - h^{-1}w^2 h_\xi\Big]\mathbf{n} \\ &+\Big[uv_\xi + vv_\eta + h^{-1}wv_\zeta - h^{-1}w^2 h_\eta\Big]\mathbf{b} \\ &+\Big[uw_\xi + vw_\eta + h^{-1}ww_\zeta + h^{-1}w(uh_\xi + vh_\eta)\Big]\mathbf{t}.\end{aligned} \tag{8}$$

The equation $\nabla \cdot \mathbf{u} = 0$ becomes

$$\frac{\partial hu}{\partial \xi} + \frac{\partial hv}{\partial \eta} + \frac{\partial w}{\partial \zeta} = 0 \tag{9}$$

Finally, the pressure term is $\nabla p = (p_\xi, p_\eta, h^{-1}p_\zeta)$. Taking the density of the fluid to be unity, and setting $u = U + u'$, the equations of motion become

$$u'_{t0} + u'u'_\xi + vu'_\eta + (1 + h^{-1})wu'_\zeta - h^{-1}w^2h_\xi + p_\xi + U_t = 0, \tag{10}$$

$$v_{t0} + u'v_\xi + vv_\eta + h^{-1}wv_\zeta - h^{-1}w^2h_\eta + p_\eta = 0, \tag{11}$$

$$w_{t0} - u'U_\zeta + u'w_\xi + vw_\eta + h^{-1}ww_\zeta + h^{-1}w(uh_\xi + vh_\eta) + h^{-1}p_\zeta - UU_\zeta = 0, \tag{12}$$

$$\frac{\partial hu'}{\partial \xi} + \frac{\partial hv}{\partial \eta} + \frac{\partial w}{\partial \zeta} + Uh_\xi = 0 \tag{13}$$

Here the subscript $t0$ indicates that ζ_0, not ζ, is held fixed.

Since $h_\eta = 0, h_\xi = -\kappa$, the equations for $\omega \equiv v_\xi - u_\eta$ and w are

$$\omega_{t0} + u'\omega_\xi + v\omega_\eta + h^{-1}w\omega_\zeta + h^{-1}\omega(u'\kappa + U\kappa - w_\zeta) - 2h^{-1}ww_\eta\kappa$$
$$-(1 + h^{-1})w_\eta U_\zeta + h^{-2}\kappa wv_\zeta + h^{-1}w_\xi v_\zeta - h^{-1}w_\eta u'_\zeta = 0, \tag{14}$$

$$w_{t0} - u'U_\zeta + u'w_\xi + vw_\eta + h^{-1}ww_\zeta - h^{-1}wu\kappa + h^{-1}p_\zeta - UU_\zeta = 0. \tag{15}$$

We rearrange these equations as follows:

$$u'\omega_\xi + v\omega_\eta = F_\omega, \tag{16}$$

$$u'w_\xi + vw_\eta = F_w, \tag{17}$$

$$u'_\xi + v_\eta = G. \tag{18}$$

Here

$$F_\omega = -\omega_{t0} - h^{-1}w\omega_\zeta - h^{-1}\omega(u'\kappa - w_\zeta) - h^{-1}\omega U\kappa + 2h^{-1}ww_\eta\kappa$$
$$+(1 + h^{-1})w_\eta U_\zeta - h^{-2}\kappa wv_\zeta - h^{-1}w_\xi v_\zeta + h^{-1}w_\eta u'_\zeta, \tag{19}$$

$$F_w = -w_{t0} + U\kappa w - h^{-1}ww_\zeta + h^{-1}wU\kappa + h^{-1}wu'\kappa - h^{-1}p_\zeta + UU_\zeta, \tag{20}$$

$$G = \frac{1}{h}U\kappa - \frac{1}{h}w_\zeta + \frac{1}{h}\kappa u'. \tag{21}$$

2.1 Ordering and expansion

We now consider the ordering of terms in the singular region $\sigma = O(1)$ as $\tau \to 0$. We have the underlying orders

$$(\xi, \eta, \zeta) = O(\tau^\gamma, \tau^\gamma, \tau^{1-\gamma}), (u', U, v, w) = O(\tau^{-\gamma}, \tau^{-\gamma}, \tau^{-\gamma}, \tau^{-\gamma}), \tag{22}$$

and also

$$\omega = O(\tau^{-2\gamma}), \kappa = O(\tau^{\gamma-1}). \tag{23}$$

Note that we have assumed the axial flow component w is of the order of the transverse components although no formal order is indicated by the motion of

C. We also have $h = 1 + O(\tau^{2\gamma-1}), t = O(\tau), p = O(\tau^{-2\gamma})$, and recall $\gamma > 1/2$. We may therefore rewrite (16),(17),(18) as follows:

$$u'\omega_\xi + v\omega_\eta = F_\omega^{(1)} + O(\tau^{-2}), \tag{24}$$

$$u'w_\xi + vw_\eta = F_w^{(1)} + O(\tau^{\gamma-2}), \tag{25}$$

$$u'_\xi + v_\eta = G^{(1)} + O(\tau^{2\gamma-2}). \tag{26}$$

Here

$$F_\omega^{(1)} = -\omega_{t0} - U\kappa\omega - w\omega_\zeta - \omega(u'\kappa - w_\zeta) + 2ww_\eta\kappa$$
$$+2w_\eta U_\zeta - w_\xi v_\zeta + w_\eta u'_\zeta = O(\tau^{-2\gamma-1}), \tag{27}$$

$$F_w^{(1)} = -w_{t0} + U\kappa w + u'U_\zeta - ww_\zeta + wu'\kappa - p_\zeta + UU_\zeta = O(\tau^{-\gamma-1}), \tag{28}$$

$$G^{(1)} = U\kappa - w_\zeta + \kappa u' = O(\tau^{-1}). \tag{29}$$

The orders of these forcing functions reflect the order of every term in the function.The left-hand sides of (24),(25),(26) are respectively $\tau^{-4\gamma}, \tau^{-3\gamma}, \tau^{-2\gamma}$, so that in each equation the ratio of forcing to left-hand side is $O(\tau^{2\gamma-1}) = o(1)$, and the ratio of the error to the left-hand side is $O(\tau^{4\gamma-2})$. Thus if δ is take as $O(\tau^{2\gamma-1})$, we may introduce expansions of the form

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 + \ldots,\ \mathbf{q}_n = (u'_n, v_n), w = w_0 + w_1 + \ldots, \tag{30}$$

$$\omega = \omega_0 + \omega_1 + \ldots,\ p = p_0 + p_1 + \ldots, \tag{31}$$

where the subscript 1 terms are smaller than the subscript 0 terms by a factor of δ, and the errors are still smaller by another factor δ.

2.2 The first-order solution and compatibility of the zeroth-order solution

The first-order terms satisfy

$$\mathbf{q}_0 \cdot \nabla[\omega_1, w_1] = [F_\omega^{(1)}(\omega_0, \mathbf{q}_0, w_0), F_w^{(1)}(\omega_0, \mathbf{q}_0, w_0)] - \mathbf{q}_1 \cdot \nabla[\omega_0, w_0], \tag{32}$$

$$\nabla \cdot \mathbf{q}_1 = G(\mathbf{q}_0, w_0). \tag{33}$$

At each section ζ =constant the zeroth-order terms represent a flow with streamlines ψ =constant, when projected onto the ξ, η plane. Here we adopt the definition of the stream function used in [Childress (1985)] because of its convenience for contour averaging: $\frac{\partial\psi}{\partial\eta} = -u'_0, \frac{\partial\psi}{\partial\xi} = v_0$. We assume now that all streamlines of interest are closed. In the special cases discussed below, the zeroth-order transverse flow consists of two families of closed streamlines bounded by a circle. We shall assume a similar topology prevails in each section, although it may not agree with the LCB couple owing to advection of vorticity by an axial flow.

We may by symmetry restrict attention to one of these closed streamline patterns. (For the LCB couple, since $U < 0$,the streamfunction lies in the interval $.67Ua < \psi < 0$ in the upper eddy and in the interval $0 < \psi < -.67UA$ in the lower eddy.)

The solution of (32) requires that the right-hand sides satisfy compatibility conditions in each region of closed streamlines. Dividing each equation by q_0 and integrating around any closed contour $\psi =$ constant we see that each right-hand side must satisfy

$$\langle RHS \rangle = 0, \tag{34}$$

where the contour integral operator $\langle \cdot \rangle$ is defined by

$$\langle \cdot \rangle = \oint q_0^{-1} \cdot ds = \oint \cdot q_0^{-2} \mathbf{q}_0 \cdot \mathbf{dx}. \tag{35}$$

The differential calculus associated with the use of this operator has been discussed in [Childress (1985)]. In that paper solutions of Euler's equations similar to those of interest here, but lacking the crucial terms associated with vortex stretching, were analyzed in a nearly two-dimensional geometry. We summarize in the appendix the results needed for the present calculation, and here state the final equations emerging from the contour averaging. The two right-hand sides in (32) yield

$$A_\psi (D_\psi \omega + U\kappa\omega) - \omega \frac{\partial(w,A)}{\partial(\zeta,\psi)} + w_\psi \int \frac{\partial(H_\psi, A)}{\partial(\zeta,\psi)} d\psi = 0, \tag{36}$$

$$A_\psi (D_\psi w - U\kappa w) + \frac{\partial(H,A)}{\partial(\zeta,\psi)} - \int \frac{\partial(H_\psi, A)}{\partial(\zeta,\psi)} d\psi - UU_\zeta A_\psi = 0. \tag{37}$$

The function $A(\psi,\zeta,t)$ is the area enclosed by the averaging contour,and $H(\psi,\zeta,t)$ is the Bernoulli function of the zeroth-order transverse low, $H = p + \frac{1}{2}q_0^2$. All spatial partials are in the independent variables ψ, ζ. The differential operator D_ψ is defined by

$$D_\psi = \frac{\partial}{\partial t}\Big|_{\xi,\eta,\zeta_0} + w\frac{\partial}{\partial \zeta} + \mathcal{V}\frac{\partial}{\partial \psi}, \tag{38}$$

where

$$\mathcal{V} = A_\psi^{-1} \langle \frac{\partial \psi}{\partial t}\Big|_{\xi,\eta\,\zeta_0} + w\frac{\partial \psi}{\partial \zeta}\Big|_{\xi,\eta,t} + \mathbf{q}_1 \cdot \nabla\psi \rangle, \tag{39}$$

have the dimensions of a velocity squared, is the product of the average contour velocity an a velocity measuring the fluid flow across the streamlines of $\mathbf{q}_0$. A expression for $\mathcal{V}$ follows from the formulas given in the appendix:

$$-A_\psi \mathcal{V} = A_{t0} + wA_\zeta + \int \frac{\partial(w,A)}{\partial(\zeta,\psi)} d\psi - U\kappa A. \tag{40}$$

2.3 Conservation of flux

We now establish the following consequence of the contour-averaged equations of the previous subsection: within the region of closed streamlines we have

$$D_\psi \Gamma = 0, \quad \Gamma \equiv \int A_\psi H_\psi d\psi. \tag{41}$$

We may choose Γ to vanish at the eddy center, so (41) holds there. We have

$$D_\psi \Gamma = \int (A_\psi H_\psi)_t d\psi + w \int (A_\psi H_\psi)_\zeta d\psi + \mathcal{V} A_\psi H_\psi. \tag{42}$$

Differentiating with respect to ψ, we get

$$(D_\psi \Gamma)_\psi = H_\psi D_\psi A_\psi + A_\psi D_\psi H_\psi + w_\psi \int (A_\psi H_\psi)_\zeta d\psi + \mathcal{V}_\psi A_\psi H_\psi. \tag{43}$$

We rewrite the ψ derivative of (40) as

$$D_\psi A_\psi = U\kappa A_\psi - A_\psi \mathcal{V}_\psi - A_\psi w_\zeta. \tag{44}$$

Using (44) and (36) with $\omega = H_\psi$ in (43), all terms cancel upon integration by parts, and (41) is established.

The physical meaning of (41) is clear. It is essentially Kelvin's theorem applied to the contours of an almost two-dimensional eddy, with Γ the circulation around a contour, expressed here as a flux integral. The $\mathcal{V}\partial/\partial\psi$ term of the differential operator D_ψ indicates that the circulation contour moving with the fluid drifts across the contours $\psi =$ constant because of the ζ dependence of w. This is an order one effect which appears with the compatibility constraints.

With (41), (37) may be rewritten

$$A_\psi [D_\psi w - U\kappa w + (H - U^2/2)_\zeta] = \Gamma_\zeta. \tag{45}$$

We thus have the four equations given by (40),(41) and (45) for the five unknowns $A, \Gamma, H, \mathcal{V}, w$. The missing relation is the solution of $\nabla^2 \psi = H_\psi$ yielding the streamlines and hence the area A contained by any contour. We are thus dealing with an example of so-called *generalized differential equations*, see e.g [Grad *et* al (1975)]. In a GDE the nonlinearities involve nonlocal dependence on the dependent variables, here the relation of streamlines to H.

A point that will be useful below concerns the ζ differentiations in (45). As written these are holding ψ fixed. The claim is that these may also be taken holding ξ, η. To prove this, note first the U^2 term depends on space through ζ only. Now

$$\frac{\partial(H,\Gamma)}{\partial\zeta}\Big|_{\xi,\eta} = \frac{\partial(H,\Gamma)}{\partial\zeta}\Big|_{\psi} + \frac{\partial\psi}{\partial\zeta}\Big|_{\xi,\eta} \frac{\partial(H,\Gamma)}{\partial\psi}. \tag{46}$$

Thus

$$A_\psi H_\zeta|_{\xi,\eta} - \Gamma_\zeta|_{\xi,\eta} = A_\psi H_\zeta|_\psi - \Gamma_\zeta|_\psi + \frac{\partial\psi}{\partial\zeta}\Big|_{\xi,\eta} [A_\psi H_\psi - \Gamma_\psi]. \tag{47}$$

Noting that $A_\psi H_\psi - \Gamma_\psi = 0$, we are done.

2.4 Equations for a singular solution

To examine blow-up in the generalized differential setting we need to scale certain variables as follows:

$$A(\zeta, \psi, t) = \tau^{2\gamma} \tilde{A}(\sigma, \psi, s), \quad H(\zeta, \psi, t) = \tau^{-2\gamma} \tilde{H}(\sigma, \psi, s), \tag{48}$$

$$w(\zeta,\psi,t)=\tau^{-\gamma}\tilde{w}(\sigma,\psi,s),\quad (\xi,\eta)=\tau^{\gamma}(\tilde{\xi},\tilde{\eta}). \tag{49}$$

Note that Γ and ψ are $O(1)$ in τ. The variable $s=\ln\tau$ is the reduced time in the evolution of the singularity.

Dropping tildes, the evolution equations in σ,ψ,s thus take the form

$$w_s+\mathcal{F}_w[x,A,\Gamma,H]=0,\quad \Gamma_s+\mathcal{F}_\Gamma[x,A,\Gamma,H]=0, \tag{50}$$

with subsidiary equations

$$\Gamma=\int A_\psi H_\psi d\psi, \nabla^2\psi=H_\psi. \tag{51}$$

We allow dependence on s which is non-exponential, so as not be equivalent to a modified power of τ.

Existence of blowup in the present setting reduces to finding a solution of this system representing a paired vortex structure which propagates with a velocity determined by the zeroth-order flow.

2.5 Symmetrization

The system just described takes on a more familiar form when expressed in an effective axisymmetric form, a process often referred to as *symmetrization*. The reduction is analogous to passage to action-angle coordinates in Hamiltonian mechanics. We first define a the effective radius of a contour $\psi=$ constant by

$$A(\psi,\zeta,t)=\pi r_e^2(\psi,\zeta,t). \tag{52}$$

Thus the symmetrized contour is a circle bounding the same area. Setting

$$\frac{\partial\psi}{\partial r_e}=v_e, \tag{53}$$

we refer to v_e as the effective θ-component of the zeroth-order velocity. We may compute in two ways

$$\frac{\partial A}{\partial r_e}=2\pi r_e=A_\psi\frac{\partial\psi}{\partial r_e}=A_\psi v_e, \tag{54}$$

and so

$$A_\psi=\frac{2\pi r_e}{v_e}. \tag{55}$$

We also set

$$D_\psi=D_e\equiv\frac{\partial}{\partial t}\Big|_{\xi,\eta,\zeta_0}+w\frac{\partial}{\partial\zeta}+u_e\frac{\partial}{\partial r_e}. \tag{56}$$

Thus u_e must be the average drift of the fluid past the contours $\psi=$ constant. From the form of D_ψ we see that the total flux across a contour must be the integral around the contour of $\mathcal{V}/\frac{\partial\psi}{\partial n}=\mathcal{V}/q_0$. If this average flux is to be given correctly by u_e we must have

$$\mathcal{V}A_\psi=2\pi r_e u_e, \tag{57}$$

or

$$\mathcal{V} = u_e v_e. \tag{58}$$

Regarded as functions of r_e, ζ, t, the conservation of mass equations now takes the form

$$\left.\frac{\partial w}{\partial \zeta}\right|_{r_e,t} + r_e^{-1}\left.\frac{\partial r_e u_e}{\partial r_e}\right|_{\zeta,t} - U\kappa = 0. \tag{59}$$

We also have, from conservation of circulation and (45)

$$D_e\Gamma = 0, \quad 2\pi r_e[D_e w - U\kappa w + (H - U^2/2)_\zeta] - v_e\Gamma_\zeta = 0. \tag{60}$$

Finally we have, from the definition of Γ (41),

$$2\pi r_e\frac{\partial H}{\partial r_e} = v_e\frac{\partial \Gamma}{\partial r_e}. \tag{61}$$

This gives us four equations for u_e, v_e, w, H, Γ, so we are missing one relation, again an indication of the generalized differential system we have. Note that, because of the differentiation property we noted above after equation (45), the ζ derivatives of H, Γ in the second of (60) may be taken at fixed r_e.

For exactly axially-symmetric flow the missing relation is

$$\Gamma = 2\pi r_e v_e, \tag{62}$$

which are easily seen to give the axisymmetric form of the first-order compatibility equations, namely (59) and

$$D_e w + \frac{\partial(p - U^2/2)}{\partial \zeta} - U\kappa w = 0, \quad D_e(r_e v_e) = 0, \quad \frac{\partial p}{\partial r_e} = \frac{v_e^2}{r_e}, \tag{63}$$

where

$$p = H - \frac{1}{2}v_e^2. \tag{64}$$

2.6 Zeroth-order solutions

The leading order terms satisfy system

$$\mathbf{q}_0 \cdot \nabla[\omega_0, w_0] = 0,\ \nabla\cdot\mathbf{q}_0 = 0, \nabla = (\partial_\xi, \partial_\eta). \tag{65}$$

These equations describe a steady two dimensional Euler flow with three velocity components. Recall we define the streamfunction ψ so that $-\psi_\eta = u_0', \psi_\xi = v_0$. Then $\omega_0 = \psi_{\xi\xi} + \psi_{\eta\eta} = H_\psi$ where $H = p_0 + \frac{1}{2}q_0^2$, is the Bernoulli function.

From (65) we see that ω_0 and w_0 are functions of ξ, η through the streamfunction ψ, with ζ, t present as free parameters. At this stage we are free to chose the dependence on ψ. Note that ψ may depend arbitrarily upon ζ and t.

We choose these functions now to conform to the assumed motion of the center vortex $C(t)$. If, as we have assumed, the center vortex $C(t)$ is moving at a given ζ with velocity $u = U(\zeta, t)$, then an observer moving with the curve

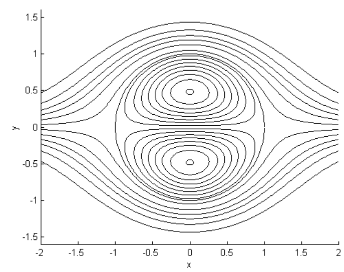

Figure 1: Streamlines of the LCB couple.

will see a velocity "at infinity" given now by $(u, v) = (-U, 0)$. Accordingly we adopt a steady flow which takes on locally the required velocity at infinity. For the moment we disregard the global restrictions on the cocoon imposed by the coordinate system and consider the local flow as two-dimensionally infinite in extent.

A sample flow is the *LCB couple* , shown in figure (1). In polar coordinates it has the stream function

$$\psi = U(\zeta, t)\sin\theta \begin{cases} r - a/r & \text{if } r > a, \\ CJ_1(Kr) & \text{if } r < a. \end{cases} \tag{66}$$

The vorticity is given by

$$\omega = \begin{cases} 0 & \text{if } r > a, \\ -K^2\psi & \text{if } r < a. \end{cases} \tag{67}$$

The paired vortex structure is here contained within the circle of radius $a(\zeta, t)$ The parameter $K(\zeta, t)$ is chosen to make $J_1(Ka) = 0$, the smallest value $Ka = 3.83$ being adopted here to give the streamline pattern shown in figure (1). The parameter $C(\zeta, t)$ is chosen to make the velocity continuous on $r = a$,

$$C = 2/[KJ_0(Ka)]. \tag{68}$$

The LCB couple corresponds to the generating Bernoulli function $H(\psi) = \pm K^2\psi^2/2$. It is clear from the contour averaged equations that the function H cannot be stipulated to remain that of the LCB couple, because of the advection of circulation by the axial flow. As a result the propagation velocity of the pair will be altered by the axial flow, and the kinematic picture of I will be expelled, raising the possibility that singularity formation is arrested.

However we shall argue here that the Lamb-Chaplygin-Batchelor solution is one of a family of propagating paired vortex structures, any one of which is a candidate for our 3D construction. Moreover, this family includes thin, distinct paired vortex tubes whose cross-sections are nearly circular, and for which the propagation velocity decouples from the internal circulation profile,

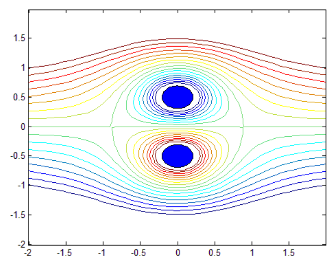

Figure 2: Streamlines of the paired vortex structure described by (136) and (137), $2\pi UL = \Gamma, R = .2L$. The vortices are shaded.

approximately. In this limit the vorticity is confined to symmetrically placed cores of order R say, placed a distance L apart. If this separation o scales persists, the pair propagates essentially as paired line vortices. In a frame stationary with the advancing pair, one sees a region of fluid which is carried with the pair and where the flow is irrotational, see figure 2. The structure of the cores can be studied analytically for small $\epsilon = R/L$, and we show in appendix B that through order ϵ^3 the boundary of the a vortex core relative two its center is given by

$$r/L = \epsilon + \epsilon^3 \cos 2\theta. \tag{69}$$

Within the core we have to this approximation

$$\psi = -\frac{\Gamma}{2\pi}\Big[\frac{J_0(kr)}{kRJ_1(kR)} + \epsilon^2 \frac{J_2(kr)}{J_2(kR)} \cos 2\theta\Big], \tag{70}$$

where $kR = 2.4048...$ is the first zero of J_0. The velocity of the pair is then given by

$$U = -\frac{\Gamma}{2\pi L}. \tag{71}$$

We now propose to treat each then vortex as circular, in which case the symmetrized model acquires a new symmetry and becomes solvable in classical terms.

3 Development of the s- independent singularity in a symmetrized model

We consider a paired vortex structure modeled in each section by two circular vortex patches of radius $R_0(\zeta, \tau)$, see figure (2). We look for a fixed point of the system, i.e. we assume no dependence upon the local time s. In the upper disc, we have the following problem. Let $r_e = \tau^\gamma R, w = \tau^{-\gamma} W, r_e v_e = \Gamma, u_e = -r_e U_t U^{-1} + \tau^{\gamma-1}\mathcal{U}$. We also now take $L = \lambda\tau^\gamma$. Then we have

$$\gamma W + (1+\gamma)\sigma W_\sigma - (1+\gamma)\sigma g_\sigma g^{-1} R W_R$$

$$+g^{-2}WW_\sigma + \mathcal{U}W_R + g^{-2}P_\sigma + 2(\gamma + (1+\gamma)\sigma g_\sigma g^{-1})W = 0, \tag{72}$$

$$(1+\gamma)\sigma\Gamma_\sigma - (1+\gamma)\sigma g_\sigma g^{-1}R\Gamma_R + g^{-2}W\Gamma_\sigma + \mathcal{U}\Gamma_R = 0, \tag{73}$$

$$P_R = \Gamma^2/R^3, \tag{74}$$

$$g^{-2}W_\sigma + R^{-1}(R\mathcal{U})_R = 0. \tag{75}$$

We require that, on $R = R_0(\sigma)$, we have

$$\Gamma(R_0, \sigma) = 2\pi\lambda g(\sigma). \tag{76}$$

The issue is, does a solution to this problem exist? We need to solve for $W, \Gamma, P, \mathcal{U}$ as functions of σ, R, for which we have four equations, then solve for $R_0(\sigma)$ to satisfy the last condition.

4 A one-dimensional model of the thin vortex

One-dimensional models of thin vortices with axial flow have proven extremely useful for analyzing vortex dynamics, see e.g. [Moore & Saffman (1972)]. We therefore turn now to such a model in the context of paired vortices. In the thin-tube limit, the vortices may be treated separately as filaments carrying constant circulation. The variation of cross-sectional area with axial flow has been discussed in a one-dimensional setting by [Lundgren & Ashurst (1989)], and our model will be closely related to theirs, but with emphasis now on the terms due to the motion and metric of the coordinate system.

The one-dimensional model of the symmetrized vortex with circular streamlines involves an axial flow $w(\zeta, t_0)$, a cross-sectional area $a(\zeta, t_0)$, and a "pressure force" $f(\zeta, t_0)$. (We adopt these symbols locally for this model only). We have the equations

$$a(w_{t_0} + ww_\zeta) + f_\zeta - 2U\kappa wa = 0, \quad a_{t_0} + (wa)_\zeta - U\kappa a = 0. \tag{77}$$

The "2" appearing in the first equation, in place of the "1" in (63) comes from the way the continuity equation is used in the derivation. The pressure force increment df is an average of all pressure forces on a small piece of the tube. Within the tube, the pressure satisfies

$$\frac{\partial p}{\partial r} = v_\theta^2/r. \tag{78}$$

If one integrates this over the cross section, using the leading term of (70), the net force on the section has the form Γ^2 times a function of kR, that is to say times a constant. That this is generally true follows from a simple dimensional argument. As the tubes stretch, the pressure distribution is determined solely by the local core area A and the total circulation Γ of the section. Dimensionally the, $p = \Gamma^2 A^{-1}$ times a function of $\xi/\sqrt{(A)}, \eta'\sqrt{A}$. Integrating of the core gives a multiple of Γ^2.

Thus there is know contribution from internal pressure forces on the ends of the piece of tube to the force increment df. As was noted by [Lundgren & Ashurst (1989)], the only contribution to df comes from the pressure force on the side wall of he piece of tube. Integrating (78) from R to ∞ we obtain

$$df = -\frac{\Gamma^2}{8\pi^2 R^2} da = -\frac{\Gamma^2}{8\pi} d \ln a. \tag{79}$$

Thus the axial momentum equation is

$$w_{t_0} + w w_\zeta + \frac{\Gamma^2}{8\pi}(1/a)_\zeta - 2U\kappa w = 0. \tag{80}$$

We now pass to the similarity form of our singular flow, assuming no variation of the variables with s. We then have, if $a = v^{-1}$, and we set $v = \tau^{-2\gamma} V(\tau, \sigma), w = \tau^{-\gamma} W(\tau, \sigma)$,

$$(2\gamma V + \mu\sigma V_\sigma) g^2 + W V_\sigma - V W_\sigma - 2(\gamma g^2 + \mu\sigma g g_\sigma) V = 0, \tag{81}$$

$$(\gamma W + \mu\sigma W_\sigma) g^2 + W W_\sigma + \frac{\Gamma^2}{8\pi} V_\sigma + 4(\gamma g^2 + \mu\sigma g g_\sigma) W = 0. \tag{82}$$

4.1 Analysis of the model

We first recall the asymptotics of g as established in I. [2] For large σ,

$$g \sim [\sqrt{2\mu}\sigma]^{\frac{-\gamma}{1+\gamma}} - \frac{\gamma}{2\mu}[\sqrt{2\mu}\sigma]^{-(\frac{2+\gamma}{1+\gamma})} + \dots. \tag{83}$$

For small σ,

$$g \sim 1 - \gamma(1+\gamma)\sigma^2 + \dots. \tag{84}$$

Since the tube must have area at $\sigma = 0$ we may assume $\lim_{\sigma\to 0} V = V_0 > 0$. The, from (81), assuming V, W are analytic in σ at $\sigma = 0$, and that V has only even powers of σ in its power series, we see that

$$V_0 W_\sigma \sim 4\gamma(1+\gamma)(1+2\gamma)\sigma^2 V_0, \tag{85}$$

so that , since necessarily $W(0) = 0$,

$$W \sim 4\gamma(1+\gamma)(1+2\gamma)\sigma^3/3, \sigma \to 0. \tag{86}$$

It is interesting that this flux is *positive*, away from the developing singularity, indicating that the squeezing down by stretching is beating the effect of pressure forces near $\sigma = 0$.

It now follows from (86), (82), and the expansion of g that

$$\frac{\Gamma^2}{8\pi} V_\sigma \sim -(8\gamma + 3)(4\gamma)(1+\gamma)(1+2\gamma)\sigma^3. \tag{87}$$

[2] The case we treat here for the function g is $\beta = 2, A = 1, \mu = 1 + \gamma$. To restore the A in what follows one need only replace σ by $A\sigma$. This step will be needed to maintain correct dimensionality.

Thus

$$V \sim V_0 - \frac{8\pi}{\Gamma^2}(8\gamma+3)\gamma(1+\gamma)(1+2\gamma)\sigma^4, \tag{88}$$

so that tube area is increasing as σ increases from 0.

Turning now to the behavior for large σ, in order that the asymptotic form of the vorticity becomes independent of time we need $W \sim c_1 g, V \sim c_2 g^2, \sigma \to \infty$ at least in leading order, $g \sim O(\sigma^{-\frac{\gamma}{1+\gamma}})$. Let us say that a term has *falloff* k if it is proportional to σ^{-k} to leading order. Then the terms $WV_\sigma - VW_\sigma$ in (81), taken together, has falloff $\frac{4\gamma+1}{\gamma+1}$, while the last term in (81) has falloff $\frac{4\gamma+2}{\gamma+1}$. Thus the latter term is asymptotically negligible. Therefore $(2\gamma V + \mu\sigma V_\sigma)g^2$ must have falloff $\frac{4\gamma+1}{\gamma+1}$, implying that

$$V \sim c_1\sigma^{-(\frac{2\gamma}{\gamma+1})} + c_2\sigma^{-(\frac{2\gamma+1}{\gamma+1})} + \dots. \tag{89}$$

Similarly the V_σ term in (82) has falloff $\frac{3\gamma+1}{\gamma+1}$, and the last term of (82) has falloff 2, and this is greater than $\frac{3\gamma+1}{\gamma+1}$ since $\gamma < 1$. Thus the latter term is negligible. Suppose now that the first term of (82) is also negligible. Then we would have

$$\frac{1}{2}W^2 + \frac{\Gamma^2}{8\pi}V \sim 0, \sigma \to \infty. \tag{90}$$

This is impossible since *area* cannot be negative. Thus the falloff of V_σ must equal the falloff of the first term of (82). This requires

$$W \sim d_1\sigma^{-(\frac{\gamma}{\gamma+1})} + d_2\sigma^{-1} + \dots. \tag{91}$$

Note that in each case the following term is smaller by the factor $\sigma^{\frac{-1}{\gamma+1}}$.

Then (82) yields

$$\frac{(1+\gamma)^{\frac{1}{1+\gamma}}}{\gamma 2^{\frac{1+2\gamma}{1+\gamma}}} d_2 + d_1^2/2 + \frac{\Gamma^2}{8\pi}c_1 \sim 0, \tag{92}$$

implying $d_2 < 0$.

4.2 Numerical solution

The one-dimensional numerical problem which must be solved to insure the existence of the needed fixed point of the generalized system will now be summarized. The system is most conveniently solved using g in place of σ as the independent variable. Also $\Gamma^2/8\pi$ may be removed by $V \to 8\pi V/\Gamma^2$, equivalent to setting $\Gamma = \sqrt{8\pi}$ leaving $V_{g=1}$ as the only parameter. The the equations are then

$$\frac{dV}{dg} = (W + \mu\sigma g^2)FV/D + GVW/D, \quad \frac{dW}{dg} = -FV/D + (W + \mu\sigma g^2)GW/D, \tag{93}$$

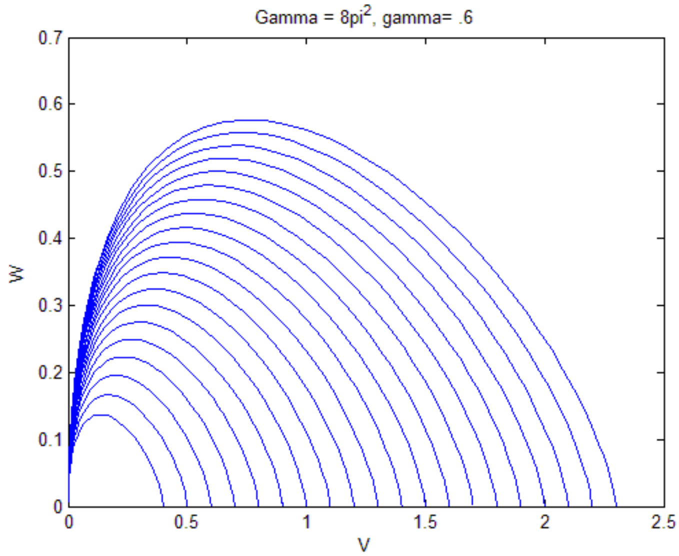

Figure 3: The solutions of (93), (98) for various V_0, $\gamma = .6$.

where

$$\sigma(g) = \frac{1}{\sqrt{2\mu}} g^{-\mu/\gamma} \sqrt{1 - g^{2/\gamma}}, \tag{94}$$

$$F = 2\mu\sigma g, \quad G = -5\gamma g^2/g' + 4\mu\sigma g, \tag{95}$$

$$g'(g) = \frac{-2\gamma\mu\sigma g^3}{1 + 2\sigma^2\mu^2 g^2}, \tag{96}$$

$$D = (W + \mu\sigma g^2)^2 + V. \tag{97}$$

The boundary conditions are

$$V(0) = W(0) = 0, V(1) = V_0 > 0, W(1) = 0. \tag{98}$$

The necessary integral curves are easily found numerically by shooting from $g = 1$ and are shown in figure 3. Note that the axial flow is always *non-negative*, never toward the singularity. This, and the smoothness of the integral curves, insures the validity of the one-dimensional model and is strongly suggestive of a well behaved fixed point for the generalized system in the thin tube limit.

4.3 The outer potential flow

The total flow fieldin the vicinity of the vortex pair includes the component of u_e given by $-U^{-1}U_t r_e$, which must be matched with an extgernaal potential flow. The potential of the outer flow, denoted by ϕ_{out}, can be derived as follows.

In the matching region the votices a locally thin ζ, r_e may be replace by z, r, the local cylindrical coordinates. The we require

$$d\phi_{out}/dr \sim -\frac{r}{\tau}(\gamma + \mu\sigma g_\sigma/g), \quad r \to 0. \tag{99}$$

We may represent ϕ_{out} as the Laplace integral

$$\phi_{out} = \int_0^\infty e^{-kz} F(k) J_0(kr) dk. \tag{100}$$

The matching requires

$$\int_0^\infty e^{-k\zeta} k^2 F(k) dk = \frac{1}{\tau}(\gamma + \mu\sigma g_\sigma/g). \tag{101}$$

Thus

$$k^2 F(k) = \frac{1}{2\pi i\tau} \int_{\mathcal{B}} e^{k\zeta}(\gamma + \mu\sigma g_\sigma/g) d\zeta, \tag{102}$$

where $\mathcal{B}$ denotes the Bromwich path in the complex ζ plane. This can be written

$$k^2 F(k) = \frac{1}{2\pi i\tau} \int_{\mathcal{B}} e^{k\zeta} \Big[\frac{\gamma G^{2/\gamma}}{1 + \gamma - \gamma G^{2/\gamma}}\Big] d\zeta, \tag{103}$$

where $G(\zeta)$ satisfies

$$\zeta = \frac{\tau^{1-\gamma}}{\sqrt{2\mu}} \Big[G^{1-1/\gamma}\sqrt{1 - G^{2/\gamma}} - 2\int_1^G u^{-1/\gamma}\sqrt{1 - u^{2/\gamma}} du\Big]. \tag{104}$$

Two points should be noted about this outer flow: (1) We have taken the ζ-axis to be a straight line. This is valid in an intermediate region distant from the center vortex, i.e. $\sqrt{\xi^2 + \eta^2} \gg \tau^\gamma$ but at a distance small compared to $\kappa^{-1} \sim \tau^{1-\gamma}$. To study the irrotational outer flow at distances $O(\kappa^{-1})$ from the center vortex we must take into account the curvature of the center vortex. (2) The pressure contributed by the outer potential vortex , at the vortex cores, is negligible compared to that already considered in our one-dimensional model of the cores. This pressure force, coming from an integral over the external vortex velocity involving the integrand v_θ^2/r, was of order $\tau^{-2\gamma}$ By contrast the radial velocity component is $O(\tau^{\gamma-1})$ when $r \sim \tau^\gamma$, contributing only order $\tau^{2\gamma-1}$ to the pressure. Similarly the ζ component of velocity in the outer flow is $O(\tau^{3\gamma-2})$ when $r \sim \tau^\gamma$, and this is again $o(\tau^{-\gamma})$ when $1/2 < \gamma < 1$. Finally the time derivative of the potential, a final contributor to the pressure at the core vortices, is of our $\tau^{2\gamma-2}$ when $r \sim \tau^\gamma$, again being $o(\tau^{-2\gamma})$.

4.4 Construction of the singularity

We have carried out here an analysis of the asymptotics of the singular flow, and have made use of a one-dimensional model of the vortex cores. Although not a rigorous proof of blow-up, we find no inconsistencies in the present construction, which would rule out the existence of a singular flow in three dimensions. On the other hand we have not explicitly dealt with the finiteness of the dynamical cocoon, which was needed only to prevent a singularity in the coordinate system from entering the computational domain. To achieve full consistency we can, for example, impose an artifical stress-free cylindrical boundary, centered at the center vortex and of radius $\frac{1}{2}\kappa^{-1}$. Boundary conditions at this outer boundary are then those of a rigid stress-free boundary. Imposing those conditions would affect our flow by asymptotically negligible amounts as $\tau \to \infty$ in the neighborhood of the dynamic cocoon at distances small compared to κ^{-1}. The singularity is thus asymptotically exact even with a proper outer boundary. One can imagine a procedure of approximation of the exact initial condition for the blow-up, as follows. With the cocoon imposed let the flow proceed as far into the singularity as desired. Then remove the cocoon boundary, add a smooth extension of the outer flow to fill all space, and use this as an initial condition for a reverse flow. That is, reverse all velocities and run forwards in time for the same elapsed time as in the approach to the singularity. In this way a sequence of approximate initial conditions $\mathbf{u}_k(T)$ are obtained, representing as k increases a closer and closer approach to blow-up. Because of the asymptotic consistency of the construction, we conjecture that this sequence will converge to a flow which is an example of an initial condition filling 3D space having bounded, continuous vorticity, producing blow-up in finite time.

A final bit of surgery is needed to eliminate infinite energy in the initial condition. Note that since only finite stretching is involved in the blow-up, one can say that only a finite amount of kinetic energy is associated with the fluid participating in the blow-up. At this point, however, we note that a suitable value of γ, namely $\gamma = (1 + \sqrt{19})/9 = .5954...$, may be picked so that three copies of the singularity may be arranged so that the center vortices will match with the sides of an equilateral triangle. Then in the neighborhood of each vertex we have a copy of the self-similar singular flow, and finite total energy of the system of three participating singular regions.

5 Discussion

5.1 General remarks

The thrust of the present paper amounts to the proposition that the present construction yields an acceptable example of Euler blow-up in three dimensions. It is acceptable in the sense that only a finite amount of vortex stretching occurs during the blow-up. Other analytical approaches suffer from the fact that infinite kinetic energy occurs *in the flow comprising the set of singular points*. Such infinite-energy solutions tell us nothing about the behavior of Euler flows of finite kinetic energy in a bounded domain.

The present part represents the second step toward verification of our proposition. It falls short of a complete verification, even in the weak sense of formal theoretical fluid dynamics, in that we have utilized a one-dimensional model of tube dynamics when in fact the problem is two-dimensional. However the robustness of 1-D theory for slender vortex tubes of near circular cross section makes for a stronger case than might be otherwise accepted. There is no indication that the 2D analysis would go much further that verifying the correctness of the one-dimensional approximation.

We have also used the asymptotic form of the thin-tube LCB solution rather than an exact form. The issue of existence of a full family of LCB-like Euler flows, and the size of this family, is an interesting question in itself, which to our knowledge has not been answered.

5.2 Stability of the singular flow

We do not consider here in any detail the important question of the stability of the proposed singular flow. Like most Euler flows, it is likely to be unstable. If this were indeed the case, then these singularities are in principle unobservable even in a perfect fluid, a point that renders computational approaches of doubtful use. Three-dimensional instability of paired vortices has been observed experimentally by [Leweke & Williamson (1998)]. The modes of instability they find appear to have their origin in the three-dimensional elliptic instability of 2D vortex cores, see [Pierrehumbert (1986), Bayly (1986)], which makes the instability sensitive to stretching of the votices. Instability in the zeroth-order solution here means that the growth is on a time scale $t^*(\tau)$ such that $dt^*/d\tau = \tau^{-2\gamma}$, the poser being that needed to make τ derivatives in the momentum equation of the same order as the advective acceleration. Thus $t^*(\tau) = \tau^{1-2\gamma}$. Thus, since $\gamma > 1/2$, $t^* \to \infty$ as $\tau \to 0$. Thus the instability has "infinite time" to grow, with complete disruption of the singularity the inevitable result. This underscores the importance of looking for the "fixed point solution", by which we mean a steady zeroth-order Euler flow together with independence of the dynamic variables on the logarithmic time scale s.

Associated with the question of instability is the issue of the "density" of initial conditions producing blow-up in some function space of allowable initial conditions. On interesting prospect is that both issues may involve the downstream $\times$-type neutral (stagnation) point of the paired vortex flow. The outer streamline boundaries of the closed region are stable manifolds of this neural point. Any perturbation of the manifolds preserving the volume of the closed region leads to mixing of fluid interior and exterior to these manifolds. Eventually these disruptions would extract vorticity from the cores and break up the pair, see figure 4. The perturbations of the manifold would be introduced by perturbations of the vortical cores, see figure 4. Note that this is a purely 2D instability of the zeroth-order flow, and is insensitive to stretching.

The "neutral-point instability" just described would explain the difficulty in identifying a blow-up initial condition , as follows. Let us introduce a measure on the set of initial condition defined by the boundaries of the vortical cores. The singular conditions, i.e. the "fixed point" contours will lie between to circles centered at the vortex center. Any other simple closed contour around the inner boundary of this annulus constitutes a possible initial condition when extended

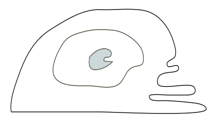

Figure 4: A possible instability of the zeroth-order solution. Only the upper vortex is shown.

in our construction along the center vortex. Now there will generally be a set of fixed point contours, parametrized by the area enclosed. But for each such area, these is at the very minimum a one (real) parameter family of contours, given say by taking the maximum excursion radially from the corresponding fixed point contour. A measure on all contours will therefore leave the fixed-point contours as measure zero. Thus the contours leading to blow-up in our construction constitute a set of measure zero in the space of feasible contours.

5.3 The impossibility of the singularity in viscous flows

The important question of singularities of solutions of the Navier-Stokes equations leads, in the present construction, to a negative result. The question is intriguing nonetheless, because the viscous term $\nu\nabla^2\mathbf{u}$, which is dominated here by $\nu(\mathbf{u}_{\xi\xi}+\mathbf{u}_{\eta\eta})$, has the same order as the acceleration in our zeroth-order solution. However this 2D Navier-Stokes equation can have no *steady* solution in finite space. Thus there would have to be evolution on the time scale t^* just introduced. The ultimate exponential decay occuring in the Stokes limit then translates to super-exponential decay in τ, overwhelming the algebraic growth associated with the singularity of vorticity.

This conclusion could not be drawn if γ were to equal $1/2$, but in that case the Euler singularity cannot occur at least according to the present construction. Nevertheless we suggest that viscous flows with structure approaching that of the present Euler flows with $\gamma = 1/2$ would be able to amplify vorticity significantly, before saturating and decaying.

5.4 Comparison of this work with the construction of Siggia and Pumir

In [Pumir & Siggia (1990)] a theory of singularity formation based upon self-stretching of paired vortices is developed. The pioneering construction proposed by Pumir and Siggia is similar in many respects to the present one. However there are important differences. In the first place, their proposal corresponds to our case $\gamma = 1/2$. It is for that reason that both Navier-Stokes and Euler flows are considered in [Pumir & Siggia (1990)]. According to [Chae (2006)] this is an example of self-similarity, where in fact an Euler singularity cannot exist. The problem one encounters when $\gamma = 1/2$, as indeed [Pumir & Siggia (1990)] knew, is that one could not guarantee the integrity of the vortex cores at the singularity time. For example, paired vortices with radius of curvature κ are drawn to one another in the direction of the binormals with a velocity proportional to $\kappa \ln(\kappa/a)$ where a is a core radius of the vortex. In the present construction this velocity is $O(\tau^{\gamma-1})$ and is negligible compared to the $O(\tau^{-\gamma})$ velocity of the vortex itself, but in [Pumir & Siggia (1990)] these velocities are comparable, with the possibility of core deformation as the vortices collide.

The main motivation of the present work is the kinematic singularity presented in I, and Pumir and Siggia seem to have not been aware of that solution of the equations of a line moving by the normal. However they do undertake local analysis of the collapse to the singularity using line vortices, and arrive at a similar conclusion of formation of a point singularity with finite total stretching.

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A Contour averaging

The function $A(\psi, \zeta, t)$ is the area enclosed by a contour have streamfunction value $\psi(\xi, \eta, \zeta, t)$ at a fixed section $\zeta =$ constant of the dynamic cocoon. Using the contour average (35), we first establish the following properties:

$$\langle 1 \rangle = A_\psi, \quad \langle \psi_\zeta \rangle = -A_\zeta, \quad \langle \psi_t \rangle = -A_t. \tag{105}$$

The first expression follows from

$$\delta A = A_\psi \delta\psi = \oint \delta n ds = \delta\psi \oint \frac{\delta n}{\delta \psi} ds = \delta\psi \oint q_0^{-1} ds = \delta\psi \langle 1 \rangle. \tag{106}$$

The second (and third) of (105) follow from differentiation of ψ for fixed ψ and s:

$$\frac{\partial \psi}{\partial \zeta}\Big|_{\psi,s} = 0 = \frac{\partial \psi}{\partial \zeta}\Big|_{n,s} + \psi_n \frac{\partial n}{\partial \zeta}\Big|_{\psi,s}. \tag{107}$$

Taking the contour average then gives

$$0 = \langle \psi_\zeta \rangle + \oint n_\zeta ds = \langle \psi_\zeta \rangle + +A_\zeta, \tag{108}$$

establishing the result. Also we see that

$$\langle \psi_\xi \rangle = \langle \psi_\eta \rangle = 0. \tag{109}$$

Indeed,

$$\oint (u_0, v_0)/q_0 ds = \oint (dx, dy) = (0, 0). \tag{110}$$

We also observe the following useful identity:

$$\int\int (\cdot) dA = \int \langle\cdot\rangle d\psi. \tag{111}$$

This follows from $dA = dsdn = dsd\psi/q_0$.

The differential operator $D \equiv \partial_t + w\partial_z + \mathbf{q}_1 \cdot \nabla$ occurs in the right-hand side of (32) and we need to compute the contour average of its action on ψ. (Here and elsewhere we have written ∂_t for ∂_{t0}.) We have, using (105)

$$\begin{aligned}\langle D\psi\rangle &= \langle\psi_t\rangle + w\langle\psi_\zeta\rangle + \langle\mathbf{q}_1 \cdot \nabla\psi\rangle,\\ &= -A_t - wA_\zeta + \oint \mathbf{q}_1 \cdot \mathbf{n} ds,\\ &-A_t - wA_\zeta - \int\int [w_\zeta\big|_{\xi,\eta} - U\kappa] dA.\end{aligned} \tag{112}$$

Using the chain rule and (105) we obtain

$$\begin{aligned}\int\int w_\zeta\big|_{\xi,\eta} dA &= \int \langle w_\zeta\big|_\psi + \psi_\zeta\big|_{\xi,\eta} w_\psi\big|_\zeta\rangle d\psi,\\ &= \int (w_\zeta A_\psi - A_\zeta w_\psi) d\psi = \int \frac{\partial(w,A)}{\partial(\zeta,\psi)} d\psi.\end{aligned} \tag{113}$$

Combining these,

$$-\langle D\psi\rangle = \int_0 [A_{\psi t} + wA_{\psi\zeta} + A_\psi w_\zeta - U\kappa A_\psi] d\psi = A_\psi \mathcal{V}. \tag{114}$$

We now indicate the derivation of (37) from the contour average of the w-part of (32). (The derivation of (36) is similar.) Now the terms of F_w^1 involving u' vanish by (109). The differentiation of w divides by the chain rule into differentiations at fixed ψ and w_ψ times differentiations of ψ. The latter are evaluated in terms of $\langle D\psi\rangle$. The only term needing discussion is $\langle p_\zeta\rangle$. We write $p = H - \frac{1}{2}q_0^2$. Then, as in

$$\int\int H_\zeta\big|_{\xi,\eta} dA = \int \frac{\partial(H,A)}{\partial(\zeta,\psi)} d\psi. \tag{115}$$

To evaluate $\langle\mathbf{q_0} \cdot \mathbf{q}_0\big|_{\xi,\eta}\rangle$ we observe

$$\int\int \nabla^2\psi dA = \int\int H_\psi dA = \oint \nabla\psi \cdot \mathbf{n} ds. \tag{116}$$

Thus

$$\int\int \frac{\partial H_\psi}{\partial\zeta}\Big|_{\xi,\eta} dA = \langle\mathbf{q}_0 \cdot \mathbf{q}_0\big|_{\xi,\eta}\rangle = \int \frac{\partial(H_\psi,A)}{\partial(\zeta,\psi)} d\psi. \tag{117}$$

These results combine to yield (37).

B Derivation of the first few terms of the thin-tube solution

We locate the tube centers at $z = \pm iL/2$ in a frame in which they are at rest. The velocity at infinity is $(U, 0), U < 0$, and the upper vortex has positive circulation, see figure 6. With our definition of ψ the complex potential valid in the irrotational region is

$$w(z) = \phi(x, y) - i\psi(x, y) = Uz - \frac{i\Gamma}{2\pi}\log(z - iL/2) + \frac{i\Gamma}{2\pi}\log(z + iL/2)$$

$$+\frac{A + iB}{z - iL/2} + \frac{A - iB}{z + iL/2} + O(z^{-2}). \tag{118}$$

Within the upper vortex we take

$$\psi_{in} = CJ_0(kr) + (D\cos\theta + E\sin\theta)J_1(kr) + F + \ldots. \tag{119}$$

Here $k, A, B, ..., F$ are real constants. We assume the boundary of the upper vortex is given by $|z - iL/2| = R + \alpha\cos\theta + \beta\sin\theta + \ldots$. Here θ is the polar angle with respect to the center of the upper vortex.

Now from (118) we have, near the upper vortex,

$$\psi_{out} = -U(y-L/2) - UL/2 + \frac{\Gamma}{2\pi}\ln(|z-iL/2|/L) - \frac{\Gamma(y - L/2)}{2\pi L} - \Im\frac{A + iB}{z - iL/2} + \ldots. \tag{120}$$

To make ψ_{in} = constant on $|z - iL/2| = R + \alpha\cos\theta + \beta\sin\theta$ we see that necessarily $J_0(kR) = 0$, and take kR as the first zero of this Bessel function. It then follows that, to lowest order,

$$\Gamma = -2\pi kRCJ_1(kR). \tag{121}$$

Thus, again to lowest order

$$F = \frac{\Gamma}{2\pi}\ln(R/L) - UL/2. \tag{122}$$

The second-order calculations involve $\cos\theta, \sin\theta$. To make ψ_{in} constant on the vortex boundary we must have

$$-CJ_1(kR)(\alpha\cos\theta + \beta\sin\theta) + (D\cos\theta + E\sin\theta)J_1(kR) = 0. \tag{123}$$

Thus

$$D = C\alpha, E = C\beta. \tag{124}$$

To make $\psi_{in} = \psi_{out}$ on the vortex boundary to second order, we must have

$$-UR\sin\theta + \frac{\Gamma}{2\pi R}(\alpha\cos\theta + \beta\sin\theta) - \frac{\Gamma R\sin\theta}{2\pi L} - \frac{B\cos\theta - A\sin\theta}{R} = 0. \tag{125}$$

To make ψ_r continuous on the boundary we must have

$$-UR\sin\theta - \frac{\Gamma}{2\pi R}(\alpha\cos\theta + \beta\sin\theta) - \frac{\Gamma R\sin\theta}{2\pi L} + \frac{B\cos\theta - A\sin\theta}{R} = 0. \quad (126)$$

It follows that

$$U = \frac{-\Gamma}{2\pi L}, A = -\Gamma\beta, B = \alpha\Gamma, \quad (127)$$

We see that α, β remain arbitrary, but these are perturbations association with small shifts of the center position of the vortex. We may therefore set $A = B = D = E = \alpha = \beta = 0$, and take $|z - iL/2| = R + \alpha_2\cos 2\theta + \beta_2\sin 2\theta + \ldots$, with

$$\psi_{in} = CJ_0(kr) + (D_2\cos 2\theta + E_2\sin 2\theta)J_2(kr) + F + \ldots, \quad (128)$$

$$w(z) = Uz - \frac{i\Gamma}{2\pi}\log(z - iL/2) + \frac{i\Gamma}{2\pi}\log(z + iL/2)$$
$$+\frac{A_2 + iB_2}{(z - iL/2)^2} + \frac{A_2 - iB_2}{(z + iL/2)^2} + O(z^{-3}). \quad (129)$$

Now we have the condition analogous to (123):

$$-CkJ_1(kR)(\alpha_2\cos 2\theta + \beta_2\sin 2\theta) + (D_2\cos 2\theta + E_2\sin 2\theta)J_2(kR). \quad (130)$$

Thus

$$D_2 = CkJ_1(kR)\alpha_2/J_2(kr), E_2 = CkJ_1(kR)\beta_2/J_2(kr). \quad (131)$$

Also the streamfunction near the upper vortex is now

$$\psi_{out} = -U(y - L/2) - UL/2 + \frac{\Gamma}{2\pi}\ln(|z - iL/2|/L) - \frac{\Gamma(y - L/2)}{2\pi L}$$
$$-\frac{\Gamma}{8\pi L^2}[x^2 - (y - L/2)^2] - \frac{B_2\cos 2\theta - A_2\sin 2\theta}{|z - iL/2|^2} + \ldots. \quad (132)$$

From the jump conditions at the vortex boundary we obtain the following conditions on the terms in $\cos 2\theta, \sin 2\theta$:

$$\frac{\Gamma R}{2\pi}(\alpha_2\cos 2\theta + \beta_2\sin 2\theta) - (B_2\cos 2\theta - A_2\sin 2\theta) = \frac{R^4}{8\pi L^2}\cos 2\theta, \quad (133)$$

$$\frac{\Gamma R}{2\pi}(\alpha_2\cos 2\theta + \beta_2\sin 2\theta) - 2(B_2\cos 2\theta - A_2\sin 2\theta) = -\frac{R^4}{4\pi L^2}\cos 2\theta. \quad (134)$$

Thus

$$\alpha_2 = R^3/L^2, B_2 = \frac{3\Gamma R^4}{8\pi L^2}, A_2 = \beta_2 = 0. \quad (135)$$

The second-order complex potential is therfore

$$w(z) = Uz - \frac{i\Gamma}{2\pi}\log(z - iL/2) + \frac{i\Gamma}{2\pi}\log(z + iL/2)$$
$$+i\frac{3\Gamma R^4}{8\pi L^2(z - iL/2)^2} - i\frac{3\Gamma R^4}{8\pi L^2(z + iL/2)^2} + O(z^{-3}), \quad (136)$$

and the boundary of the upper vortex is given by

$$r = R + \frac{R^3}{L^2}\cos 2\theta. \quad (137)$$

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