Topological fluid dynamics for fluid dynamicists

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1 Introduction

These notes are preliminary and informal. My aim is to provide an introduction to some of the basic ideas that have emerged in recent decades in the fluids literature that have a fundamentally topological character. That is, rather than focusing on details such as velocity profile or pressure field, this material deals with persistent invariant properties of a flow field, features which often have a geometric interpretation. A basic reference is the book of Arnold and Khesin [1]. Other references will be given below. My interpretation of TFD will omit many technical details, but will I hope serve its purpose of translating some of the important ideas into the language of classical fluid mechanics.

Topological ideas arise very naturally in fluid dynamics through the geometry of the vorticity field, and early work by Lord Kelvin and others explored knottedness of vortex tubes, for example. However the bringing together of inherently topological ideas from fluid dynamics into a coherent theory has been a fairly recent endeavor, associated with the work of Moffatt among fluid dynamicists, and Arnold and Freedman among mathematicians.

An early example of topological thinking in fluid dynamics is the idea of a simply-connected domain and the role of multiply-connected domains in the analysis of lift by an airfoil in two-dimensional flow. A simple material curve which encircles a "hole" in the domain is fundamentally distinct from one which does not, and this property is an invariant of the flow field. There is no reference here to dynamics, so this is the essence of the topological viewpoint. Similarly, by calculating the circulation on such a curve the lift of a foil may be identified with a "topological invariant". In three dimensions, the knottedness of material curves provide immediate examples of topological invariants. Perhaps the simplest example of such invariants can be seen in the advection of a scalar field c(x,y,t):

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0. \tag{1}$$

Since contours c =constant are material, there are basic features of the flow (such as the number of local maxima or minima of c) which are topological invariants (since diffusion of c is not allowed).

We may thus summarize our viewpoint here as a focus on properties of a flow field which are invariant under the flow (or are approximately invariant based on time scales of interest). Since the technical aspects of fluid dynamics, e.g. the regularity of the functions which are involved, will not be an essential part of story. We shall simply assume that all functions we consider have the necessary smoothness for the operations we perform on them. We further restrict attention to flows of a fluid of constant density in a domain $\mathcal{D}$ which is either all of R^3, or else a bounded region isomorphic to the interior of a sphere, with zero normal velocity imposed at its boundary. In a technical sense "smooth" functions and maps are where the defining functions have continuous first partial derivatives, which we adopt as its minimal meaning.

2 Preliminaries

We are interested in fluid motions in R^N, $N = 2$ or 3. But we are also interested in sets of points within R^N. A *manifold* of dimension $M \leq N$ in R^N is a set of points each of which is fixed by specifying M real numbers. Examples of manifolds in fluid dynamics are a stream-surface of a steady flow in surface in R^3 (two dimensional), and a vortex line in R^3 (one-dimensional). The above definition is however two general for our purposes. For example there exists a one-one correspondence of the interval $0 \leq x \leq 1$ and the square $0 \leq x \leq 1, 0 \leq y \leq 1$, obtained by alternating the digits of the decimal expansions of x, y to obtain a point on the line. This "one-dimensional" manifold looks like a surface. We thus demand that our manifold of dimension M look "in the small", like R^M. Thus, our curves in R^3 will look, on small scales, like little line segments, i.e. copies of a piece of R^1. This is basically demanding that the point in R^N vary smoothly with the M "coordinates" of the manifold. By saying that the manifold looks locally like R^M, we are further assuming that the fluid is indeed a continuum, i.e. expelling completely the underlying molecular structure. Locally we thus have open sets and can define continuous functions on the manifold in the usual way (the preimage of an open set is open).

A flow will distort any embedded manifold in general. This distortion can be described by a map. The maps which we shall use will be one-one, continuous, and with continuous inverse. Such maps are called *homeomorphisms*. If both the map and its inverse are differentiable, the map is called a *diffeomorphism*. If a velocity field is differentiable, it will move manifolds around in such a way that they are connected by diffeomorphisms. As we have already noted above, we shall not be discussing the degree of smoothness and so for all practical purposes we can take the diffeomorphisms we discuss to be C^∞, i.e. C^k for any positive integer k. [1] We can also use the informal terminology "smooth transformation of manifold" to describe a diffeomorphism.

As is already evident, the main kind of diffeomorphism which will interest us here is that induced by a smooth flow of a fluid. We are familiar with the Lagrangian picture of fluid flow: each fluid particle moves with position

[1] In some accounts a diffeomorphism is defined as C^∞.

$\mathbf{x}(t, \mathbf{x}_0)$ where $\mathbf{x}_0$ may be taken as the position at some time t_0. This function is determined from an Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$ by the equation

$$\frac{\partial \mathbf{x}}{\partial t}\Big|_{\mathbf{x}_0} \mathbf{x} = \mathbf{u}(\mathbf{x}(t, \mathbf{x}_0), t), \mathbf{x}(t_0, \mathbf{x}_0) = \mathbf{x}_0. \tag{2}$$

The *time-T map* of a manifold $\mathcal{M}$ contained in a fixed region D for fluid, by a flow field $\mathbf{u}$ defined over D, is obtained by turning on the flow at some time t_0 and letting it carry the points of $\mathcal{M}$ for a time T. For any differentiable $\mathbf{u}$ this map is a diffeomorphism of D into itself, and when restricted to the points of $\mathcal{M}$, defines a diffeomorphism connecting $\mathcal{M}$ and its time-T image $\mathcal{M}_T$. We shall sometime refer to the time-T map induced by a flow on a manifold as the time-T *Lagrangian map* associated with this flow.

The mathematical description of this process makes use of a different terminology. A vector field in R^3 is a special object belonging to the *tangent bundle* of R^3. The tangent bundle is actually a rather familiar object, since we like to draw a picture of a vector field by picking points of R_3 and attaching arrows to them. The tangent bundle of R^3 is R^3 with a copy of R^3 attached to each of its points. It is thus a product of the form $R^3 \times R^3$. The second R^3 is where the "arrow" resides. For a single point, the collection of all the vectors that attach to it (a copy of R^3) is the *tangent space* at that point, or alternatively the *fiber* of the tangent bundle at that point. In this terminology a vector field in R^3 is a map from R^3 into the tangent bundle, i.e. an assignment of a single 3-vector to every point of R^3.

The term "tangent" is being used here because of intuition provided by dynamics. The definition (2) of a velocity as derivative of position suggests this usage.

For a manifold $\mathcal{M} \subset R^3$ we can similarly define the tangent bundle $T\mathcal{M}$ of the manifold. Locally, $T\mathcal{M}$ looks like a piece of $R^3 \times R^3$, but it is interesting that in general we cannot assign vectors on a manifold in such a way that they vary continuously on the manifold. Such singular behavior means that we cannot in general write the manifold's tangent bundle as a product. The simplest example of this is the manifold in R^3 which is the surface of a sphere. The question is, can we assign vectors in R^3 arbitrarily to each point of the sphere while maintaining continuity? The counterexample is to take vectors which have the same nonzero length and which are everywhere tangent to the surface of the sphere. Then we have established (after a trivial dilation) a map of the surface of the sphere into itself. If this map is continuous as we assert, the Brouwer fixed point theorem asserts that there must a fixed point. But this is impossible here since the tangent and position vectors are orthogonal. This result is a special case of an interesting body of theory concerning the global properties of tangent bundles of manifolds.

A more picturesque view of this particular result is that the horizontal wind velocity on the surface of a planet must always vanish at one point (at least).

2.1 Transport of vector fields

Our object now is to discuss the action of a diffeomorphism on a vector field. Since the diffeomorphism is simply a reshuffling of points, the way in which the attached vectors change under this reshuffling is completely open. This is where the reference to "tangent" in the tangent bundle becomes important. We want to consider the mapping of vector fields which are generated in a special way, namely, as tangent vectors to a curve C, parametrized say by s: $C = \{\mathbf{g}(s)|s_1 < s < s_2\}$. Consider a point $\mathbf{g}_0 = \mathbf{g}(s_0)$ on this curve and let $\mathbf{v}_0$ be the tangent vector to C at $\mathbf{g}_0$, i.e. $\mathbf{v}_0 = \mathbf{g}'(s_0)$. A diffeomorphism $\mathbf{y} = F(\mathbf{x})$ now acts on the points of the curve. The points of the mapped curve are now $F(\mathbf{g}(s))$. We now define the transport of $\mathbf{v}_0$ to be the tangent vector of the new curve with respect to the parameter s. Thus, by the chain rule,

$$\frac{d}{ds}F(\mathbf{g}(s))|_{s=s_0} = J\mathbf{v}_0, \tag{3}$$

where $J_{ij} = \frac{\partial y_i}{\partial x_j}$ is the Jacobian of the diffeomorphism. We thus see that this kind of transport is exactly the same as the transport of what we called "material vector field" in the first semester. That is, a vector field is material if it is proportional to the differential $\mathbf{ds}$ on any field line. Material vector fields are usually referred to as *invariant* fields in the topological setting. The picturesque term is to say that the lines are "frozen into the fluid".

By the way, what we call "field lines" , i.e. the integral curves of

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \ldots = \frac{dx_N}{u_N}. \tag{4}$$

is often translated as *the one-dimensional foliation* of the vector field.

Once we have this idea of transport of a vector field, we can talk about it's action symbolically. If F is a diffeomorphism and $\mathbf{v}$ a vector field, its transport by F is written $F_*(\mathbf{v})$, and is called the *push forward* of $\mathbf{v}$. The *pull-back* of $\mathbf{v}$ is defined as the push forward of $\mathbf{v}$ under F^{-1}, and is written $F^*(\mathbf{v})$. Similar symbols apply to the a time-t diffeomorphism of a flow, usually written as ϕ_t. A material vector field $\mathbf{v}$ has the property that $\mathbf{v}(\phi_t\mathbf{x}) = \phi_{t*}\mathbf{v}(\mathbf{x})$. This is mathematically equivalent to the Cauchy representation of vorticity, for example, recall $\omega_i(\mathbf{x}) = J_{ij}(\mathbf{a}, t)\omega_{0j}(\mathbf{a})$.

2.2 Lie derivatives

Lie derivatives are operations involving two vector fields. Consider the vector fields $\mathbf{u}, \mathbf{v}$, both of which are independent of time. Think of these as *steady flows*. Let the flow $\mathbf{u}$ have the time-t map ϕ_t. The *Lie derivative of* $\mathbf{v}$ *with respect to* $\mathbf{u}$ is defined as

$$\mathcal{L}_\mathbf{u}\mathbf{v} = \frac{d(\phi_t^*\mathbf{v})}{dt}|_{t=0}. \tag{5}$$

Note that this involves the pull back. There are two things going on here, the flow $\mathbf{u}$ is moving the base points, and the attached vectors are being transported. This operation in Euclidean space is illustrated in the following diagram:

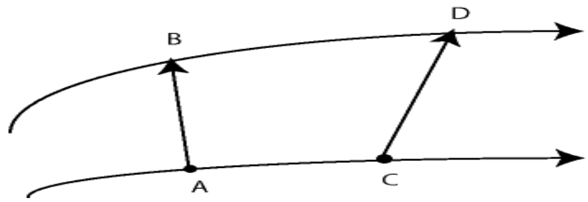

Figure 1: Computing the Lie derivative. The curved lines are streamlines of the flow $\mathbf{u}$.

The straight vectors are differential elements $\mathbf{dg}$, see text. The point C is the basepoint $\mathbf{x}$ and $CD = \mathbf{dg}(\mathbf{x})$ is the vector observed at this base point at time t. Since we will consider push forward transport rather than pull back, we are going to compute the negative of the Lie derivative. Now $AB = \mathbf{dg}(\mathbf{x}-\mathbf{u}(\mathbf{x})dt)$, and by vector transport we have $AC = \mathbf{u}(\mathbf{x})dt$ and $BD = \mathbf{u}(\mathbf{x}+\mathbf{dg}(\mathbf{x}))dt$. Then by vector algebra

$$\mathbf{dg}(\mathbf{x})|_{t+dt} = \mathbf{dg}(\mathbf{x} - \mathbf{u}(\mathbf{x})dt) + \mathbf{u}(\mathbf{x} + \mathbf{dg}(\mathbf{x}))dt - \mathbf{u}(\mathbf{x})dt. \tag{6}$$

Thus

$$\mathbf{dg}(\mathbf{x})|_{t+dt} - \mathbf{dg}|_t = [-\mathbf{u} \cdot \nabla \mathbf{dg} + \mathbf{dg} \cdot \nabla \mathbf{u}]dt. \tag{7}$$

With $\mathbf{dg}/ds = \mathbf{v}$ we get the result that the Lie derivative is the rate of change of the vector $\mathbf{v}$ observed to be occuring at the point $\mathbf{x}$ under reversal of the flow field $\mathbf{u}$. As a direct calculation using our symbols,

$$\begin{aligned} \frac{d}{dt}\Big[\frac{\partial \phi_t^* x_i}{\partial x_j} v_j(\mathbf{x})\Big]_{t=0} &= \Big[\frac{\partial \phi_{-t} x_i}{\partial x_j}\Big]_{t=0} u_k \frac{\partial v_j}{\partial x_k} + v_j \frac{\partial}{\partial x_j}\Big[\frac{\partial}{\partial t}\phi_{-t} x_i\Big]_{t=0} \\ &= \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u}. \end{aligned} \tag{8}$$

The Lie derivative is alternatively written as the *Lie bracket* $[\mathbf{u}, \mathbf{v}]$. Note that $[\mathbf{v}, \mathbf{u}] = -[\mathbf{u}, \mathbf{v}]$. The Lie bracket also satisfies the *Jacobi identity*

$$[[\mathbf{u}, \mathbf{v}], \mathbf{w}] + [[\mathbf{v}, \mathbf{w}], \mathbf{u}] + [[\mathbf{w}, \mathbf{u}], \mathbf{v}] = 0. \tag{9}$$

Example 1 *In steady Euler flow of a fluid of constant density* $[\mathbf{u}, \omega] = 0$*, where* $\omega = \nabla \times \mathbf{u}$ *is the vorticity field. The idea of a fluid being "frozen into the fluid" involves Lie differentiation. Minus the Lie derivative (i.e. the observed change of the vector at a point* $\mathbf{x}$*, must then be in fact equal to the partial derivative with respect to time of the of the (in general time-dependent) vector field at the point* $\mathbf{x}$*. Thus* $\mathbf{v}(\mathbf{x}, t)$ *is frozen into a fluid with velocity field* $\mathbf{u}(\mathbf{x}, t)$ *provided that* $\mathbf{v}_t + [\mathbf{u}, \mathbf{v}] = 0$*. Note that time is a parameter in the flows in the Lie bracket. The Lie derivative is computed for each time.*

Problem 1 *Verify* (9).

2.3 Lie Groups

A Lie group is a set of smooth transformations of a smooth manifold which satisfy the group properties: (i) For every two transformations f, g the composition $f \circ g$ (where first g, then f is applied) is in the set; (ii) For every f in the set, the inverse transformation f^{-1} is in the set. Thus every Lie group contains the identity transformation e which does nothing to the manifold.

The tangent space for the manifold may here be described as the tangent space of the group evaluated at the identity, and is called *the vector space of the Lie algebra* associated with the group. Equipping this vector space with the Lie bracket $[\mathbf{u}, \mathbf{v}]$, we obtain the *Lie algebra* of the Lie group.

Example 2 *The Lie group corresponding to rigid rotation of bodies in R^3 is the rotation group SO(3). The corresponding Lie algebra is that of the skew-symmetric 3×3 matrices. Such matrices define the angular momentum vector. The corresponding Lie derivative is the cross product of these associated vectors.*

Example 3 *The volume-preserving diffeomorphisms of a given fluid domain D form a Lie group, denoted by $\mathcal{D}$. The vector space of the Lie algebra are the divergence-free vector fields in D.*

3 Helicity and the topology of vortex lines and tubes

In this section we use fluid dynamics language to discuss an interesting topological invariant. We consider first an fluid of constant density filling R^3. We assume that the velocity and vorticity fields at some given time have the property that the quantity

$$\mathcal{H} = \int_{R^3} \mathbf{u} \cdot \omega \, dV < \infty, \ \omega = \nabla \times \mathbf{u}. \tag{10}$$

We call $\mathcal{H}$, following Moffatt [4], the (total) *helicity* of the flow. This quantity is also know (for its earlier introduction by Hopf) as the Hopf invariant.

Theorem 1 *If the fluid moves according to Euler's equations for a perfect fluid, then, provided fields decay sufficiently fast at ∞, the helicity is a constant of the motion.*

To prove this, we compute over a sphere $S \subset D$

$$\frac{\partial \mathcal{H}}{\partial t} = \int_S \mathbf{u}_t \cdot \omega + \mathbf{u} \cdot \omega_t. \tag{11}$$

Now

$$\mathbf{u}_t = -\nabla h + (\mathbf{u} \times \omega), \ h = p + \frac{1}{2}u^2, \tag{12}$$

and

$$\omega_t = \nabla \times (\mathbf{u} \times \omega). \tag{13}$$

Using $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$, we obtain

$$\frac{\partial \mathcal{H}}{\partial t} = \int_S \nabla \cdot [-\omega h + ((\mathbf{u} \times \omega) \times \mathbf{u}] dV, \tag{14}$$

and so

$$\frac{\partial \mathcal{H}}{\partial t} = \int_{\partial S} [\frac{1}{2}(u^2 - p)\omega - \mathbf{u} \cdot \omega \mathbf{u}] \cdot \mathbf{dS}. \tag{15}$$

As S grows to infinity we obtain necessary conditions on the decay of the fields for invariance to hold. If the fluid is contained in a region D bounded by a rigid wall ∂D, it follows that $\mathbf{u} \cdot \mathbf{dS} = 0$ but it is also necessary that $\omega \cdot \mathbf{dS}$ vanish there, i.e. that the vortex lines not penetrate the boundary. This is not in general a condition that will be satisfied at a rigid boundary.

Suppose now that we consider a subset $V \subset D$ which moves with the fluid. Let $\mathcal{H}_V$ be the helicity within this region. Then

$$\frac{d\mathcal{H}_V}{dt} = \int_V \nabla p \cdot \omega + (\omega \cdot \nabla \mathbf{u}) \cdot u dV = \int_V \nabla \cdot [(u^2/2 - p)\omega] dV, \tag{16}$$

We see that if $\omega \cdot \mathbf{dS} = 0$ instantaneously at the boundary of V, then the helicity is invariant. This is the real importance of this invariant, since it says that any isolated patch of vorticity (bounded by a material surface of vortex lines) posses a helicity invariant.

Problem 2 *A divergence-free field can be defined as* $\mathbf{u} = \alpha \nabla \beta + \nabla \phi$, *where* α, β, ϕ *are smooth functions and* $\nabla \cdot (\alpha \nabla \beta + \nabla \phi) = 0$. *Show that if these functions decay sufficiently rapidly at infinity then this field has zero total helicity* $\mathcal{H}$.

3.1 Discrete vortex tubes

An important example of a simple moving region V of the kind just described is that occupied by a closed vortex tube. Taking the tube to be so thin that it may be approximated as a closed curve C carrying a fixed circulation γ (which can be measured by carrying out a circulation integral about any simple closed path C' which encircles the tube once), we see that the helicity integral here is just $\gamma \Gamma_C$, where Γ_C is the circulation integral for C, $\Gamma_C = \oint_C \mathbf{u} \cdot \mathbf{ds}$. If there is no other vorticity present, $\Gamma_C = 0$. If there are other vortex tubes, but none penetrate a surface having C as its boundary, then by Stokes theorem applied to that surface we again have $\Gamma_C = 0$. However if there is another vortex tube C' with circulation $\gamma_\prime$, which penetrates the surface associated with C, then $\Gamma_C = \gamma_\prime$ and the helicity computed over the region occupied by C is $\gamma\gamma'$.

The configuration of two closed vortex tubes shown in Figure 2 illustrates this situation. When as shown the two tubes are linked, the total helicity over both tubes is $2\gamma\gamma'$ in the case shown. I the sense of one of the linked tubes is reversed, the total helicity is $-2\gamma\gamma'$. If the tubes are not linked the total helicity is zero.

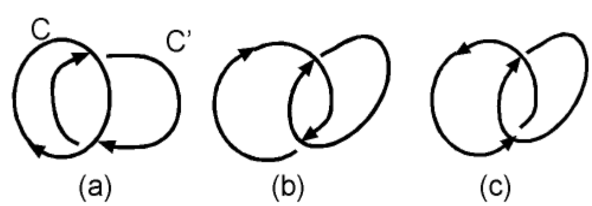

Figure 2: The helicity of the two tubes is (a) 0, (b)$2\gamma\gamma'$, (c)$-2\gamma\gamma'$. Here γ, γ' are the circulations carried by the tubes C, C'.

In general a closed vortex tube which is knotted, in the sense that it cannot be continuously deformed into a circle, can be decomposed into two or more linked but unknotted loops. We illustrate this in Figure 3, where we prove that the total helicity of the tube shown is $+2$ times the tube circulation.

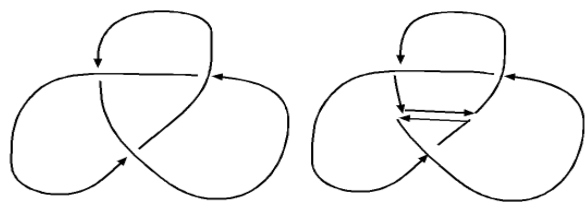

Figure 3: A knotted vortex tube has helicity twice the tube circulation.

We shall say that the two rings of Figure 2(a) are each an *unknot*. The structure in Figure 3 is a knot, since it cannot be deformed smoothly into an unknot. We say that the two unknots in Figure 2(b) are *linked*. Any knot is equivalent is equivalent as in Figure 2 to a structure consisting of linked unknots One slight ambiguity: we have to decide whether or not the reduction to linked unknots continues as in Figure 3(c) to links which encircle on once, i.e. are *simply linked*. Clearly the surgery of Figure 3(c) can proceed to a structure of simply linked rings.

Once we have this structure, we see that the helicity will have the form

$$\mathcal{H} = \sum_{ij} \gamma_i \gamma_j \alpha_{ij}, \tag{17}$$

where α_{ij}, defined as the circulation about the ith ring due to the presence of the jth ring, is ± 1. If we simply state that the structure consists of linked unknots, allowing for multiple encirclings, then the α_{ij} are nonzero integers.

Since the computation of helicity of tubes is thus reduced to computation for pairs of linked unknots, we can easily express the α_{ij} as an integral based upon the Biot-Savart law. Recall that the velocity field due to the jth ring is given by

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \oint_{C_j} \frac{\gamma_j \mathbf{t}_j \times \mathbf{y}}{|\mathbf{y}|^3} ds', \quad \mathbf{y} = \mathbf{x} - \mathbf{x}'. \tag{18}$$

Thus the circulation about the i ring may be computed from this and thus we obtain

$$\alpha_{ij} = \frac{1}{4\pi} \oint \oint \frac{\mathbf{y} \cdot (\mathbf{t} \times \mathbf{t}')}{|\mathbf{y}|^3} ds' ds. \tag{19}$$

Consider two simply-linked unknots as in Figure 2(b,c). We define the *Gauss* *linking number* of the structure by

$$GLN(K) = \frac{1}{4\pi} \oint \oint \frac{\mathbf{y} \cdot (\mathbf{t} \times \mathbf{t}')}{|\mathbf{y}|^3} ds' ds, \tag{20}$$

where the integrals are in the direction of the orientation of the tubes, s applying to one tube and s' to the other. The GLN is thus ± 1 while we have that the helicity is $\pm 2\gamma\gamma'$. Thus the GLN of a pair of simply linked unknots (one encircling) is ± 1. The GLN in general can be used to determine the number of time one closed curve winds around a second closed curve. If we consider an arbitrary structure K of vortex tubes with various knots and links, carrying various circulations, the helicity of the structure is similarly given by

$$\mathcal{H}(K) = \frac{1}{4\pi} \oint \oint \gamma(s)\gamma(s') \frac{\mathbf{y} \cdot (\mathbf{t} \times \mathbf{t}')}{|\mathbf{y}|^3} ds' ds, \tag{21}$$

but now the integrals with respect to both s and s', so structures are counted twice in the same way that the factor of 2 appeared for simple linked rings.

It is important to remark that, while the helicity is clearly saying something about the "tangleness" of the vortex tubes, it by no means provides complete information about a knot. A case in point are the "Borromean rings", see page 132 of [1]. This structure consists of three entangled rings, the helicity of any two being zero, and the total helicity of the structure is zero, yet the rings cannot be disentangled without breaking a ring. Higher order topological invariants are needed to obtain more information about the properties of a knot. There are an infinite hierarchy of topological invariants of increasing complexity, but they are not easily used in the fluid-dynamical context.

3.2 Beltrami fields

Consider a constant density fluid flow in a domain D with finite nonzero helicity $\mathcal{H}$. What is the minimum value of the integral $\int_D \omega^2 \, dV$ that such a flow can have? We first note that by the Schwarz inequality

$$[\int_D \omega \cdot \mathbf{u} \, dV]^2 \leq \int_D u^2 \, dV \int_D \omega^2 \, dV. \tag{22}$$

Now the Poincaré inequality states that for any domain D there is a length L such that $\int_D \omega^2 \, dV \geq L^{-2} \int_D u^2 \, dV$ where $\mathbf{u}$ as any flow in D satisfying $\mathbf{u} \cdot \mathbf{dS} = 0$ on ∂D. Thus we have that the helicity bounds $\int_D \omega^2 \, dV$,

$$\int_D \omega^2 \, dV \geq L^{-1} |\int_D \omega \cdot \mathbf{u} \, dV|. \tag{23}$$

This inequality becomes an equality if there exists a flow $\mathbf{u}$ contained in D such that $\omega = \lambda \mathbf{u}$, $\lambda = L^{-1}$. Flows with vorticity and velocity everywhere parallel vectors are called *Beltrami flows.* In the case here we have these vectors proportional by a constant. The eigenvalue problem of solving $\omega = \lambda \mathbf{u}$ in D

with zero normal velocity on ∂D leads to eigenvalues and the minimum of the absolute values of these eigenvalues gives L^{-1}.

Beltrami fields have many remarkable properties. Steady fields of this type, having the general form $\omega = f(\mathbf{x})\mathbf{u}, \mathbf{u} \cdot \nabla f = 0$, are incompressible Euler flows with nontrivial vorticity. They offer the only solutions which have chaotic Lagrangian particle paths, as we shall see below.

One way to construct a large class of Beltrami fields is to set

$$\mathbf{u} = \lambda \nabla \times \mathbf{A} + \nabla \times \nabla \times \mathbf{A}, \tag{24}$$

where λ is a constant. Taking the curl,

$$\nabla \times \mathbf{u} = \lambda \nabla \times \nabla \times \mathbf{A} - \nabla \times \nabla^2 \mathbf{A}. \tag{25}$$

Thus if $\nabla^2 \mathbf{A} = -\lambda^2 \mathbf{A}$, we see that $\nabla \times \mathbf{u} = \lambda \mathbf{u}$. For example in R^3, $\mathbf{A} = \mathbf{i} r^{-1} \sin(\lambda r)$ determines a Beltrami field. An example with non-constant λ is $\mathbf{u} = (\psi_y, -\psi_x, W(\psi))$ where $\psi(x, y)$ satisfies $\nabla^2 \psi = F(\psi)$. Then $\nabla \times \mathbf{u} = (W' \psi_y, -W' \psi_x, -F)$. Then, if $F = -WW'$, we obtain a Beltrami field with $\lambda = W'(\psi)$.

4 Steady Euler flows

We take the density as unity and consider the solution of

$$\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0, \tag{26}$$

in a compact domain D bounded by an analytic surface ∂D. If the first equation is written in the alternate form

$$\mathbf{u} \times \nabla \times \mathbf{u} = \nabla H, \ H = p + \frac{1}{2} u^2, \tag{27}$$

we exhibit the *Bernoulli function* $H(\mathbf{x})$. Since clearly

$$\mathbf{u} \cdot \nabla H = 0, \quad \nabla \times \mathbf{u} \cdot \nabla H = 0, \tag{28}$$

we see that *the flow lines of both velocity and vorticity line in the surfaces $H =$ constant.* Note that this statement carries weight only where such surfaces exist, i.e. H is not constant. The picture to have is that H is a smooth function which foliates the domain into surfaces of constant H, *Bernoulli surfaces* say. These Bernoulli surfaces are oriented by the vector ∇H. In the seminal paper [5], Arnold proved the following result:

Theorem 2 *Suppose that* $\mathbf{u}$ *is an analytic steady Euler flow in D which is not everywhere collinear with its curl. Then the field lines of the flow lie either in invariant tori, or else on invariant surfaces diffeomorphic to an annulus. On the tori the flow lines are either all closed or all dense, and on each annulus all flow lines are closed.*

The two types of fluid structures are typified by nested tori on which flow lines are twisted about without being knotted, and annular motions in which every fluid particle moves on a circular orbit about a fixed axis of symmetry (but keep in mind we admit diffeomorphic equivalents to these field lines). This is a remarkable theorem, because if you think about an arbitrary domain it would seem very difficult to get a flow decomposable in this way to fit inside it and still satisfy (26). Apart from questions of stability, which we disregard here, the matter at hand is whether or not a solution of (26) tends to have a structure foliated by the Bernoulli surfaces.

The exceptional case H =constant of course can occur, leading to Beltrami flows! But we can equally well question the relevance of Beltrami flows to solutions obtained in a compact domain. However it is interesting that for the question of global regularity of solutions of the IVP, the possible regularity is thought to be linked to the *depletion of nonlinearity* caused by $\mathbf{u} \times \nabla \times \mathbf{u}$ becoming small in (27).

Even if $\nabla \times \mathbf{u} = \lambda \mathbf{u}$ we necessarily have $\mathbf{u} \cdot \nabla \lambda$ and so also $\lambda^{-1}(\nabla \times \mathbf{u}) \cdot \nabla \lambda = 0$, so if $\lambda(\mathbf{x})$ foliates D into surfaces λ = constant, we are back the previous situation. Thus actually the only alternative to the structures of theorem (2) is the case λ = constant. (The existence of regions of this kind in real Euler flows, where velocity and vorticity are locally proportional by a constant, is controversial.)

Nevertheless flows of this kind, i.e. where $\mathbf{u}$ is an *eigenfunction of the curl operator*, are easily constructed. The method described above is one example of a class of these flows. Another approach is to look for spatially periodic flows in R^3, i.e. flows on the torus T^3, of this kind. An interesting example is the family of ABC flows,

$$\mathbf{u} = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x). \tag{29}$$

The most symmetric case is $A = B = C = 1$. This flow is replete with invariant tori, which thread through a lattice of stagnation points (where $\mathbf{u} = 0$). However near the boundaries of these foliated regions there is a region of finite volume in which a single field line is dense on T^3. If the square of one of the parameters exceeds the sum of the squares of the other two $A^2 > B^2 + C^2$ for example, the stagnation point vanish and the region of chaotic behavior becomes much larger.

Problem 3 *Using Matlab ODE routines compute the Lagrangian trajectories of an ABC flow, and plot the intersections of orbits with a plane x, y, or $z = 0$ mod 2π. Try various starting points with $A^2 > B^2 + C^2$, and also with $A = B = C = 1$.*

4.1 An extremal property

The steady Euler flows in D are extremals of the following minimization problem: Find the minimum kinetic energy

$$E = \frac{1}{2}\int_D u^2 dV \tag{30}$$

obtained from a given flow $\mathbf{u}$ by the action of volume–preserving diffeomorphisms on of the domain D.

To prove this, observe that the infinitesimal diffeomorphism defined by a divergence–free vector field η will act to change $\mathbf{u}$ by an amount

$$\delta\mathbf{u} = [\eta, \mathbf{u}] = \eta\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\eta = \nabla\times\mathbf{u}\times\eta \tag{31}$$

Thus

$$\delta E = \int_D \mathbf{u}\cdot[\nabla\times\mathbf{u}\times\eta]dV, \tag{32}$$

but this can be written as

$$\begin{aligned}\delta E &= \int_D \big[-\nabla\cdot[\mathbf{u}\times(\mathbf{u}\times\eta)] + \nabla\times\mathbf{u}\cdot(\mathbf{u}\times\eta)\big]dV \\ &= \int_D \big[-\nabla\cdot[\mathbf{u}\cdot\eta\mathbf{u} - u^2\eta] + \eta\cdot[(\nabla\times\mathbf{u})\times\mathbf{u}]\big]dV.\end{aligned} \tag{33}$$

The first term integrates to zero because of the tangency condition on ∂D. Since η is an arbitrary volume–preserving diffeomorphism, the extremal flows must be such that $(\nabla\times\mathbf{u})\times\mathbf{u}$ is the gradient of some function, $-H$ say, and thus must satisfy $[\mathbf{u}, \omega] = 0$, and we are done.

We remark that in this first variation of $\mathbf{u}$ we can think of $\delta\mathbf{u}$ as $t\mathbf{u}$ for small t. Higher variations come from solving

$$\frac{d\mathbf{u}}{dt} = [\eta, \mathbf{u}] \tag{34}$$

as a formal power series in t, $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) + t\mathbf{u}_1 + (t^2/2)\mathbf{u}_2 + \ldots$. Thus $\mathbf{u}_1 = [\eta, \mathbf{u}]$ and $\mathbf{u}_2 = [\eta, \mathbf{u}_1] = [\eta, [\eta, \mathbf{u}]]$.

In particular we can compute the second variation of E as

$$\delta^2 E = \int_D \big(|\delta\mathbf{u}|^2 + \mathbf{u}\cdot\delta^2\mathbf{u}\big)dV \tag{35}$$

We insert $\mathbf{u}_1 = \delta\mathbf{u}$ and $\mathbf{u}_2 = \delta^2\mathbf{u}$ to obtain, after use of the divergence theorem

$$\delta^2 E = \int_D \big([\eta, \mathbf{u}]\cdot[\eta, \mathbf{u}] + (\eta\times\mathbf{u})\cdot[\eta, \mathbf{u}]\big)dV. \tag{36}$$

The second variation is of importance in discussing stability of steady flows.

4.2 Two dimensions

In R^2 the energy minimization becomes a minimization of

$$E = \int_D \psi_x^2 + \psi_y^2 \, dS \tag{37}$$

where ψ is the stream function; the extremals then satisfy Euler's equations in the form $\nabla\psi \times \nabla\nabla^2\psi = 0$. Thus a problem of the form $\nabla^2\psi = F(\psi)$ arises, stating that a fixed value of the vorticity can be associated with any given streamline. A wide variety of such flows presumably exist, and can in principle be obtained by energy minimization. Choose an initial streamfunction ψ_0 of desired topology. Take for example the flow within a circular region with streamlines as in Figure 4(a), not an Euler flow. We want to obtain an Euler flow from it by minimizing energy, subjecting the flow to area–preserving diffeomorphisms.

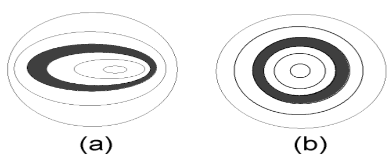

Figure 4: Energy minimization with one critical point of ψ.

We shall prove that this minimization will lead to concentric circular streamlines with the area of the region between streamlines with any two values of ψ being preserved.

We will show this by induction from the critical point outwards. Near the critical point of the starting flow, the streamfunction must have the form $\psi = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ bounding an area πab, a, b being small, on the boundary of which $\psi = 1$ (this value being immaterial). Let the values of a, b after minimization be a', b'. The contribution to E within this region is easily calculated to be $\pi(b/a + a/b)$, which is minimized by taking $a' = b' = \sqrt{ab}$ to preserve area.

Now assume that the regions bounded by two neighboring streamlines are examined in sequence, building up a circular "onion" out to the pair shown bounding the shaded region of Figure 4(a). We want to show that the contribution to the energy from this region will simply add another layer to our circular onion. Applied to this region, our diffeomorphism will preserve area, and the object is to minimize energy. Let the values of the bounding streamlines be ψ and $\psi + \Delta\psi$. Since $\Delta\psi =$ constant, we see that the width of the layer, $\Delta n(s)$ say, satisfies $q\Delta n = \psi =$ constant where $q^2\psi_x^2 + \psi_y^2$ is the square of the speed.

Thus the constancy of area under the diffeomorphism requires that

$$\oint \Delta n ds = \Delta\psi \oint \frac{ds}{q} \tag{38}$$

be constant. Subject to this constraint, we wish to minimize the contribution to E, given by

$$\Delta E = \oint q^2 \Delta n ds = \delta\psi \oint q ds. \tag{39}$$

Introduction a constant Lagrange multiplier μ , we minimize

$$\oint (\frac{1}{q} - \mu q)\ ds, \tag{40}$$

and obtain $q =$ constant, hence $\Delta n =$ constant, hence we get another annular layer to our onion, and we have established that the minimizing pattern is that of Figure 4(b).

We interject here a comment regarding the role of small diffusion in the 2D case and in particular for the topology just described. In questions of topology of steady inviscid flows there naturally arises the question of whether or not the pattern obtained is "stable" under the addition of small viscosity. By this we mean to imply that the Euler flow is a close approximation to some solution of the Navier-Stokes equations if the viscosity is sufficiently small, and that this viscous solution should not be topologically distinct except perhaps is thin boundary layers. We refer here simply to existence of solutions, which may or may not be physically stable steady flows. There is thus a hierarchy to consider: Stable NS limits $\subset$ NS limits $\subset$ Euler minimizers.

Consider an arbitrary simply-connected domain in 2D and let the streamfunction have a single critical point as in Figure 4(a). After energy minimization to an Euler flow, we have a similar topology, with vorticity now constant on every streamline. What will small viscosity do to this flow? One problem here that, if the non-slip condition is no applied to the fluid at ∂D, the flow will simply decay to zero. Hence to pose the question we need to maintain the flow and we will assume that the speed of the minimizer on ∂D is applied as a boundary value for the Navier-Stokes flow. It then turns out that the topology does not change in this case, but there is a dynamical effect on the vorticity, on a time scale of order L^2/ν where L is a diameter of D. Prandtl argued that ultimately the Euler flow must one of globally constant vorticity. To see this, we take the resulting steady solution of the Navier-Stokes equations (assuming its existence, and disregarding stability issues) and consider

$$I_\psi = \int_{D_\psi} \nabla \cdot [\omega \mathbf{u} - \nu \nabla \omega] dS = 0, \tag{41}$$

where D_ψ is the region within streamline with streamfunction value ψ. Since $\mathbf{u}$ is tangent to ∂D_ψ we see that necessarily

$$\nu \oint_{\partial D_\psi} \frac{\partial \omega}{\partial n} ds = 0. \tag{42}$$

Now we are assuming very small viscosity, but the factor ν here can nevertheless be dropped. On the other hand we expect the Navier-Stokes solution to be close to an Euler flow, in the sense that $\omega \approx \omega(\psi)$. Thus (42) becomes

$$\nu \oint_{\partial D_\psi} \frac{\partial \omega}{\partial n} ds \approx \frac{d\omega}{d\psi} \oint_{\partial D_\psi} q ds = 0, \tag{43}$$

and the only reasonable way to achieve this is to set $\frac{d\omega}{d\psi} = 0$.

This is an interesting argument, which is far from being a rigorous proof, but is seems to be well supported by numerical studies. The "theorem" ws rediscovered by Batchelor in [6] and "Prandtl-Batchelor" theory provides techniques for selecting among candidate minimizers, those Euler flows which are in fact formal asymptotic approximations to steady solutions of the Navier-Stokes equations at large Reynolds number. Further application of this theory is described in [7].

What happens under minimization for 2D topologies more complex than that of Figure 4? Imagine the level lines of a smooth ψ function with an arbitrary landscape

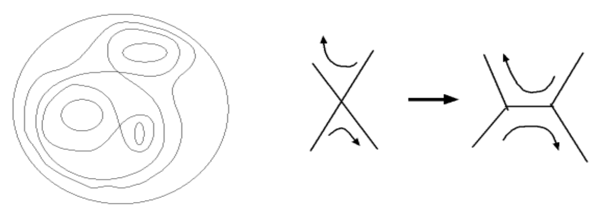

Figure 5: Energy minimization with $\times$-points.

In general the minimization will not preserve the smoothness. We see in Figure 5 how an $\times$-type critical point can split to bring oppositely direction velocity into contact at a discontinuity. The role of small viscosity in smoothing this discontinuity is obviously an important issue, and again the question must be studied of the survival of the topology in the Navier-Stokes problem.

4.3 Minimization in 3D

4.3.1 An analogous magnetostatic problem

We want to consider now a more general fluid allowing for conduction of electric currents. We take the fluid to be a "perfect" conductor, meaning that there is no dissipation involved with the flow of electric currents. We assume however that the fluid is viscous. Our aim is introduce a class of static magnetic fields that are mathematically equivalent to the class of steady Euler flows (for comparable boundary conditions, as we shall see). These magneto-static fields will be obtained under the condition that the fluid be a rest.

The equations of magnetohydrodynamics (MHD), under the condition of perfect conductivity, are

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p - \mu\nabla^2\mathbf{u} = \mathbf{J}\times\mathbf{B},\ \mathbf{J} = \nabla\times\mathbf{B}, \tag{44}$$

$$\mathbf{B}_t = \nabla\times(\mathbf{u}\times\mathbf{B}), \tag{45}$$

$$\nabla\times\mathbf{u} = \nabla\times\mathbf{B} = 0. \tag{46}$$

Here $\mathbf{B}$ is the magnetic field and $\mathbf{J}$ the current field. We contain this fluid with a volume D with

$$\mathbf{u} = 0, \mathbf{B}\cdot\mathbf{N} = 0,\ \mathbf{x}\in\partial D. \tag{47}$$

The latter condition states that magnetic field lines do not penetrate the boundary ∂D.

If we dot (44) with $\mathbf{u}$ and dot (45) with $\mathbf{B}$ and add the two equations, some vector manipulation, the divergence theorem, and (46) lead to

$$\frac{d}{dt}(E_K + E_M) = \Phi, \tag{48}$$

where $E_K = \frac{\rho}{2}\int_D u^2 dV$ is the total kinetic energy, $E_M = \frac{1}{2}\int_D B^2 dV$ the total magnetic energy, and Φ is the rate of viscous dissipation in D.

Problem 4 *Verify* (48)*.*

Moffatt [8] has argued as follows: since the total energy is decreasing so long as $\mathbf{u}\neq 0$, if the total energy is known to be bounded from below, we are assured that it must tend to a limit for large times. Assume for the moment that the process terminates with $\mathbf{u} = 0$. In this case we are left with a solution to the magnetostatic problem

$$\nabla p + \mathbf{B}\times(\nabla\times\mathbf{B}) = 0, \nabla\cdot\mathbf{B} = 0. \tag{49}$$

This is analogous to an Euler flow under the identification of $-p$ with the Bernoulli function H, and $\mathbf{B}$ with the velocity field. (It is interesting that (45) states that "$\mathbf{B}$ is frozen into the fluid" and so in this equation $\mathbf{B}$ is analogous to the vorticity. The steady analog is different from that suggested by the behavior of the fields under diffeomorphisms.)

Moffatt then proposes to exploit this to obtain steady Euler flows of non-trivial topology, by imposing an initial magnetic field with non-zero total helicity $\mathcal{H}$. We have seen above that under this condition E_M is indeed bounded below. Thus the above mimimization process should be accessible.

Problem 5 *Show that a similar approach to energy minimixation is possible if* (44) *is replaced by*

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p + k\mathbf{u} = \mathbf{J}\times\mathbf{B}, \tag{50}$$

where k is a positive constant. This correspond to assuming the fluid moves in a porous medium with porosity k.

Thus the idea is that the total energy continues to drop as the fluid moves a reshuffles the magnetic field lines. In view of (45), the topology of the lines is preserved since they move with the fluid. Any excess magnetic energy should be converted to kinetic energy to move the lines toward a configuration of lower E_M. Eventually the fluid comes to rest with the magnetic field in an "Euler flow" configuration with minimal energy.

While this scenario is appealing, it masks a number of very important mathematical difficulties. The main problem is that there is no control over the smoothness of the liming magnetic field. The discontinuity emerging in Figure 5 is a pale shadow of what can happen with the chaotic structure allowed in 3D. It is conceivable that the limiting $\mathbf{B}$ is differentiable almost nowhere. For example, take an initial condition that has some chaotic structure but is not a Beltrami field. If the limiting field obtained under the relaxation of the flow were smooth, we know it must decompose into Arnolds's tori and annuli. But this is impossible because of the chaotic structure. Then if smooth it would have to be a smooth Beltrami field with λ = constant, e.g. an ABC flow, but such solutions are rare and are not likely to be compatible with boundary conditions.

Thus we have a huge gap in the theory. On the other hand we can expel this problem by admitting as suitable weak solutions all limits obtained under the relaxation process. It is in this sense that Moffatt's idea can survive. The topology of the initial condition is retained, but any reasonable kind of smoothness is lost.

This may be an acceptable mathematical framework in which to construct weak steady solutions of Euler's equations, but it is catastrophic from the physical standpoint. The reason is the loss of topological structure which would occur in the presence of viscosity, irrespective of how small it may be. In the magnetic context, the same can be said with the introduction of small electric conductivity. (It that case (45) is modified by a term exactly analogous to the viscous term in the vorticity equation, see [7].)

We remark that there is an interesting physical problem where these ideas appear to be very relevant– namely in the flares and microflares of the solar magnetic field. We summarize the phenomena here; some further details are given in [9]. Solar flares are rapid discharges occurring in the upper solar photosphere, a layer thickness 500 km, and above that in the lower chromosphere, a layer of thickness 2.5×10^6 m. Above that is the solar corona, extending far out into space and notable for the location of solar prominences. Solar flares are believed to be associated with magnetic reconnection and hence directly related to the change of field topology. One of the major unexplained properties of solar structure is the high temperature of the corona, $\sim 10^6\ ^oK$, compared with $\sim 4300\ ^oK$ in the photosphere . Eugene Parker has proposed that the heating is largely due to *microflares* involving rearrangement of magnetic topology. The prominences may be viewed as arched tubes of magnetic flux, extending far into the corona but with their ends anchored in turbulent fluid of high electrical conductivity. As the field lines are entangled by the fluid motions at their points of attachment, the tubes try to relax to a minimum energy configuration, much as in Moffatts's relaxation problem, and if it is hypothesized that the resulting minimized is non-smooth, the microflares would result from finite electrical conductivity within the coronal gas.

5 Vortex stretching and global regularity of the IVP

The question of global regularity of solution of the IVP for Euler's equations is an important mathematical topic, which is perhaps not unrelated to topological issues, a point we wish to develop in this section. In two space dimensions it is known that smooth initial data results in globally smooth solutions, see [10] p. 116. This is essentially a result of material property of the scalar vorticity in 2D. In 3D no analogous results are known, and the question of global regularity remains open.

Although there is considerable disagreement on this point, many theoretical fluid dynamicists consider "Euler blow-up" a definite possibility. By blow-up we here mean finite time blow-up, the loss of the regularity of the initial data in finite time. A crude argument for such a behavior replaces the vorticity equation in 3D,

$$\frac{d\omega}{Dt} = \mathbf{u} \cdot \nabla\omega, \tag{51}$$

by

$$\frac{D\omega}{dt} = |\omega|\omega. \tag{52}$$

The fact that $\frac{\partial u_i}{\partial x_j}$ and $\omega_i\omega_j(|\omega|^{-1}$ are totally different object notwithstanding, (52) is dimensionally consistent and remains quadratic in the derivatives of the velocity. The dot product of (52) with ω then gives the scalar equation

$$\frac{D|\omega|}{Dt} = |\omega|^2. \tag{53}$$

This has solution

$$|\omega(\mathbf{x}(t,\mathbf{a}),t) = (\frac{1}{|\omega(\mathbf{a},0)|} - t)^{-1}. \tag{54}$$

Thus, vorticity first becomes infinite if we follow the fluid particle corresponding to the initial point of maximum modulus of the vorticity, the time of blow-up being t^* = one over the modulus of vorticity at this point.

Underlying this kind of estimation is the elementary invariant scaling of Euler's equations,

$$\mathbf{u} \to U\mathbf{u}, \mathbf{x} \to X\mathbf{x}, t \to Tt/U,\ T = X/U, \tag{55}$$

which indicates that any velocity derivative scales like $1/t$.

5.1 The Beale-Kato-Majda condition

While the result (54) completely bypasses the kinematics of vorticity in 3D, it is interesting that it is consistent with one of the few rigorous constraints on blow-up which has been given, namely the Beale-Kato-Majda condition (see [11], and also [10], chapts. 3 and 5):

Theorem 3 *Suppose that t^* is the largest time of existence of a smooth solution of the IVP for Euler's equations in R^3. Then* [2]

$$\int_0^t \max|\omega|(s)ds \to \infty, \text{ as } t \to t^*. \tag{56}$$

Remark: We note in passing that, under the transformation (55), $D\mathbf{u}/Dt \to (U^2/X)D\mathbf{u}/Dt$ while the viscous term of the Navier-Stokes equation obeys $\nu\nabla^2\mathbf{u} \to (U/X^2)\mu\nabla^2\mathbf{u}$. These are the same only if $U = X^{-1}$ and so $T = X^2$. Leray long ago noted that on this last result admits Navier-Stokes solutions of the form

$$\mathbf{u} = \frac{1}{\sqrt{t^* - t}}\mathbf{U}(\mathbf{x}/\sqrt{t^* - t}), \tag{57}$$

necessarily also compatible with (56), but now suggestive of blow-up of a viscous flow. While most fluid dynamicists do not consider the blow-up of viscous flow (which would lead to infinite velocity of the above scaling were correct) as physically plausible, the mathematicians have not ruled out such a possibility.

Problem 6 *What is the form of the vorticity equation if the Leray scaling is assumed?*

The proof of (56) involves some simple and direct estimates, which we summarize below, and one very technical estimate, whose proof we shall skip. The Sobolov space $H^m(R^3)$T consists of $L^2(R^3)$ functions whose derivatives up to and included order m are also L^2. The norm $\|\mathbf{u}\|_m$ is the square root of the sum of the squares of u and all derivatives through order m. For vectors the Euclidan norm squared replaces the square.

First, it is easy to show that, for any m, an H^m Euler flow in R^3 satisies

$$\frac{d\|\mathbf{u}\|_m}{dt} \le c_m \max|\nabla\mathbf{u}|\|\mathbf{u}\|_m. \tag{58}$$

Here c_m is some constant. To see this for $m = 0$, dot the momentum equation with $\mathbf{u}$ and integrate over R^3, using the divergence theorem. The derivatives of $\mathbf{u}$ can be treated analogously, by repeated integration by parts.

Grönwall's lemma infers from (58) a bound on $\|\mathbf{u}\|_m$:

$$\|\mathbf{u}(\cdot, T)\|_m \le \|\mathbf{u}(\cdot, 0)\|_m e^{\int_0^T c_m \max|\nabla\mathbf{u}|dt}. \tag{59}$$

[2] As is shsown in [10], in the following one may substitute, in place of $\max|\omega|(t)$, either $\max_{ij}|\frac{\partial u_i}{\partial j}|(t)$ or $\max|t_i t_j \frac{\partial u_i}{\partial j}|(t)$, where $\mathbf{t}$ is the unit direction vector of the vorticity field.

Turning now to the vorticity equation, a differential inequality is readily derived for the L^2 norm of the vorticity, $\|\omega\|_0$. Applying Grönwall's lemma to it yields the inequality

$$\|\omega(\cdot, T)\|_0 \leq \|\omega(\cdot, 0)\|_0 e^{\int_0^T c \max |\nabla \mathbf{u}| dt}. \tag{60}$$

A very technical potential theory estimate then bridges the gap between $\max |\nabla \mathbf{u}|$ and $\max |\omega|$:

$$\max |\nabla \mathbf{u}| \leq C(1 + ln^+ \|\mathbf{u}\|_3 + ln^+ \|\omega\|_0)(1 + \max |\omega|), \tag{61}$$

where $ln^+ x$ vanishes for $x < 1$. See [10] for the derivation of (61). Now we insert (59) with $m = 3$ and (60) into (61) to obtain

$$\max |\nabla \mathbf{u}|(t) \leq C_1 \big[1 + \int_0^t \max |\nabla \mathbf{u}(\cdot, s)| ds\big](1 + \max |\omega(\cdot, t)|). \tag{62}$$

Again applying Grönwall's lemma,

$$1 + \int_0^t \max |\nabla \mathbf{u}(\cdot, s)| ds \leq 2e^{C(t + \int_0^t \max |\omega(\cdot, s)| ds)}. \tag{63}$$

Returning now to (59) and using (63)

$$\|\mathbf{u}(\cdot, T)\|_m \leq \|\mathbf{u}(\cdot, 0)\|_m e^{c_m [2e^{C(t + \int_0^t \max |\omega(\cdot, s)| ds)} - 1]}. \tag{64}$$

Thus we bet a bound on the H_m norm so long as ${}^{C(t + \int_0^t \max |\omega(\cdot, s)| ds)}$ is bounded.

We have looked here at the details of ths BKM condition in order to emphasize that the mathematics uses the Euler equations only to obtain simple energy estimates of derivatives of velocity. (the estimate (61) applies to any divergenc-free vector field and its curl.) All of the complexity of the vorticity kinematics embodied in the Biot-Savart Law is expelled from consideration, so that one is working rigorously at a level of apprxomation analogous to (53). It is therefore perhaps not too surprising that the BKM condition is consistent with (54).

The question we address here is, what is the next level of approximation (above simple energy estimates) that might provide enough of a description of the vorticity to settle the matter of global regularity. It is possible that the nest level is one of topological constraints, a possibility that accounts for inclusion of this topic in these notes.

5.2 Infinite kinetic solutions exhibiting singularities

If you believe in Euler blow-up, it is comforting to find supporting solutions, however trivial. The simplest one I know of is

$$\mathbf{u} = A\mathbf{x}/(t^* - t), \tag{65}$$

where A is any constant symmetric matrix with zero trace. The corresponding pressure, making this an Euler flow, is

$$p = -\frac{1}{2}\mathbf{x} \cdot (A + A^2) \cdot \mathbf{x}/(t * -t)^2, \tag{66}$$

In R^3 this flow has infinite kinetic energy for all time, and blows up everywhere simultaneously. Also, clearly solutions of this type can be potential flows (A a diagonal matrix for example) and so have zero vorticity! Moreover (see below) they also exist in two-dimensions, where we know regularity prevails!

While such solutions of this kind will never settle the regularity issue, they have generated a certain amount of interest. The reason for this is that more structure can be introduced so that the vortical field becomes more interesting, and while the kinetic energy remains infinite the hope is raised that the solutions might be a "local" approximation to a singularity occurring in a Euler flow of finite kinetic energy.

For example, J.T Stuart [12] introduced a class of solutions of the form

$$(u, v, w) = (f(x, t), yg(x, t), zh(x, t)). \tag{67}$$

A related two-dimensional example [13] is

$$(u, v) = (f(x, t), -yf_x(x, t)). \tag{68}$$

We will here focus on (68) since it illustrates most easily the way these simple solutions mimic 3D vorticity dynamics. Since $\psi = yf(x, t)$ is the streamfunction for this flow, we may imagine a flow in the semi-infinite channel $0 < x < L, y > 0$, and to make the boundary of the channel a streamline we impose

$$f(0, t) = f(L, t) = 0. \tag{69}$$

We also impose the initial condition $f(x, 0) = f_0(x)$. Putting (68) in the 2D vorticity equation we obtain

$$\mathcal{L}f_{xx} = 0 \tag{70}$$

where the operator $\mathcal{L}$ is defined on any function $q(x, t)$ by

$$\mathcal{L}q = q_t + [f, q]. \tag{71}$$

Here $[f, q] = fq_x - qf_x$ is the Lie bracket. We also introduce a 1D Lagrangian variable $x(t, a)$,

$$\frac{dx}{dt} = f(x, t), \quad x(0, a) = a, \tag{72}$$

and let $J = \frac{\partial x}{\partial a}$ be its Jacobian. Differentiating (72) we have, if $\dot{q} = q_t + fq_x$, $\dot{J} = f_x J$, i.e. $\mathcal{L}J = 0$. Then (70) may be "solved" in the form

$$f_{xx}(\mathbf{x}(t, \mathbf{a}), t) = f_0''(a)J(a, t). \tag{73}$$

Note this is analogous to the Cauchy solution of the vorticity equation. There is also an analog to the Biot-Savart integral, as we now show. Multiply (73) by J, integrate with respect to a, and set $s = 2f_0'(a)$ to obtain

$$2f_x = \int J^2 ds. \tag{74}$$

Note that now the initial function has been absorbed into a new independent variable, s. Now $f_x = \dot{J}/J$, and if if we define $\frac{1}{2}\dot{H} = \dot{J}/J$ we obtain the IVP

$$H_{st} = e^H, \ H(s,0) = 0. \tag{75}$$

This is Liouville's equation, a general solution of which can be given in closed form, see [14]. However, we need only find a *particular solution* of this equation. To insure that our transformation $a \to s$ is 1-1 we now make the technical assumption that $f_0'' \neq 0$, although this is not actually necessary.

Thus for our purposes it is sufficient to note that $\mathcal{L}F_{xx} = 0$ is solved by the time-independent function $f = A \sin Bx$ for any constants A, S. With this f we may evaluate J and express the answer in terms of s to obtain

$$J = [\cosh(kt) - (s/2k)\sinh(kt)]^{-1}, \ k = AB. \tag{76}$$

Problem 7 *Verify* (76).

In order to satisfy the boundary conditions on f we need an arbitrary function of time, not present in (76). However we can use the following fact: if $\tilde{H}(s,t)$ is a solution of Liouville's equation, then so is $H(s,\tau) = ln(\dot{\tau}(t)) + \tilde{H}(s,\tau)$ for any $\tau(t)$. We thus obtain the general solution

$$J = \frac{\dot{\tau}^{\frac{1}{2}}}{\cosh(k\tau) - (s/2k)\sinh(k\tau)}. \tag{77}$$

Since $\int_0^L J da = L$ we obtain a differential equation for $\tau(t)$ in the form

$$\dot{\tau}^{\frac{1}{2}} \int_0^L \frac{da}{\cosh(k\tau) - (s(a)/2k)\sinh(k\tau)} = L. \tag{78}$$

The choice of k is immaterial so long as this last integral exists. Once τ is found, (77) may be integrated to find f_x and then f.

As an example consider

$$f_0 = \frac{1}{2}a(2-a), \ L = 2. \tag{79}$$

There results

$$f(x,t) = \Big[\frac{2(e^{2\tau} - e^{(2-x)\tau}}{\tau(e^{2\tau} - 1)} - \frac{x}{\tau}\Big]\Big(\frac{\sinh\tau}{\tau}\Big)^2, \tag{80}$$

$$t(\tau) = \int_0^\tau u^2/\sinh^2(u)du. \tag{81}$$

Blow-up occurs when $\tau = \infty$ and thus where

$$t = t^* = t(\infty) = \int_0^1 \frac{\ln v}{v-1}dv = \frac{1}{6}\pi^2. \tag{82}$$

5.3 The search for a nontrivial singularity

The infinite energy solutions are sufficiently suggestive to have spawned many numerical experiments of the 3D IVP. One example with an interesting set of symmetries is the initial velocity,

$$\mathbf{u} = [\sin y + \sin z, \sin z + \sin x, \sin x + \sin y]. \tag{83}$$

The most symmetric candidate that has been examined is axisymmetric flow with swirl, i.e. velocity fields of the form $(u_z(r,z), u_r(r,z), u_\theta(r,z))$. If the flow is confined to a thin annulus $R < r < R+\Delta r$, Euler's equations may be shown to reduce to those of Boussinesq convection in two dimensions [15]. In Boussinesq convection the fluid flow is driven by small density variations which do not affect the divergence-free property of the velocity. The density perturbation is however a material scalar. Thus if ρ is the perturbation around a density of 1, the equations of Boussinesq convection are (in a unit gravitational field)

$$D\rho/Dt = 0, \quad D\omega/Dt = -\rho_x, \tag{84}$$

where $\mathbf{u} = (\psi_y, -\psi_x), \omega = -\nabla^2\psi$. Again, for these models there has been active computational experimentation, but all of the results claiming evidence of singularity formation are subject to resolution problems. It is not easy to establish the existence of a singularity by accurate computation.

Other approaches have focused on the self stretching of paired vortex tubes of opposite sign. Each vortex tube advects the other, and both can be stretched in a 3D setting. Here the resolution problems persist, but also the finite size of the vortex tube is a problem, and as the singularity is approached the treatment of the tube as a line breaks down. Renormalization methods have been tried in these cases, but then the problem is losing control over the macroscopic dynamics, even at the level of maintaining conservation of energy.

I have digressed on these analytic issues mainly to suggest that it could well be that all evidence of singularity formation is based on an incomplete representation (analytical or geometrical) of the vorticity structure. We therefore will look below at the stretching of vortex filaments as an exercise in geometry, with the aim of perhaps linking the global regularity of Euler flows to the topology of the vorticity field.

In this regard we mention some interesting recent results of Constantin *et al.* concerning the vorticity structure needed to develop a singularity in finite time [16]. They show that if the unit vector giving the vorticity direction is sufficiently well behaved near the regions of largest vorticity, then a singularity cannot form.

5.4 Local induction for a thin vortex tube

Before considering the interaction of distinct vortex tubes, we consider the motion induced by a thin vortex tube on itself. Let the radius of curvature of the tube be of order L, and consider the tube to have a cross-sectional radius

$\epsilon \ll L$. Then there is a dominant effect of the tube which can be estimated, with the result that the velocity V orthogonal to the osculating plane, i.e. in the direction of the binormal, becomes infinite as $\epsilon/L \to 0$, has an expansion of the form

$$V \sim \frac{\gamma\kappa}{4\pi}\log(\epsilon/L) + O(1) \tag{85}$$

in this limit, here γ being the circulation about the vortex and κ the curvature. Although the expansion can be made sensible with a careful modeling of the tube, the fact is that a "line vortex" of fixed circulation is not a sensible object the moment it ceases to be absolutely straight, i.e. to be the 3D extension of the point vortex of two dimensions. It follows from the fact that the divergent term is in the direction of the binormal, that *to the first approximation self-induction of a thin tube does not stretch the tube.*

This property of vortex tubes it not significant for continuous distributions of vorticity, wherein the field may be regarded as consisting of tubes of infinitesimal strength. Applied to finite strength tubes some approximation is needed, the so-called *local induction hypothesis*, the and the resulting theory of self-induction of a line vortex, not involving any line stretching, has received considerable attention.

5.5 A regularity result in 3D

Some 3D motions are sufficiently symmetric to obtain global regularity in a manner analogous to that of 2D flows. Consider a flow with cylindrical symmetry and no swirl. The vorticity is $(0, 0, \omega_\theta)$ in (z, r, θ) coordinates, and velocity is $(u_z, u_r, 0)$. Let the initial vortical field $\omega_{\theta 0}(z, r)$ be smooth, bounded, and and supported on a region of volume V_0. It follows that the support of the vorticity at any future time has volume V_0. We further assume $|r^{-1}\omega_{\theta 0}(z, r)| < c$ on its initial support.

We can estimate $\max(|\mathbf{u}|)$ over all space as follows (see [10]):

$$\max(|\mathbf{u}|) \leq \Big|\frac{1}{4\pi}\int_{|\mathbf{y}|\leq R_0}\frac{\mathbf{y}\times\omega'}{y^3}dV'\Big| + \Big|\frac{1}{4\pi}\int_{|\mathbf{y}|\geq R_0}\frac{\mathbf{y}\times\omega'}{y^3}dV'\Big| \quad , \mathbf{y} = \mathbf{x} - \mathbf{x}'. \tag{86}$$

Clearly

$$\max(|\mathbf{u}|) \leq \max_{supp}(|\omega_\theta(z, r, t)|)[4\pi R_0 + V_0 R_0^{-2}]. \tag{87}$$

If we set $R_0^3 = V_0$, we get $\max(|\mathbf{u}|) \leq c_1 \max_{supp}|\omega|$, where $c_1 = (1 + 4\pi)R_0$. Now in this Euler flow

$$\omega_\theta(z, r, t)/r = \omega(r_0, z_0, t)_{\theta 0}/r_0, \tag{88}$$

where (z, r) and is the terminal point of a fluid particle which started at (z_0, r_0). Now let $R(t)$ be the radius of the support at time t. Then we have

$$dR/dt \leq \max(\mathbf{u}) \leq c_1 \max_{supp}(\omega_\theta) \leq cc_1 R. \tag{89}$$

By Grönwall's lemma the radius of the support, hence the maximum vorticity, grows at most exponentially in time.

5.6 Interacting unknots

The previous example of global regularity applies to a flow all of whose vottex lines remain circles for all time. This geometrical constaint is what controls the solution and leads to global regularity.

We now consider some simple calculations for the stretching of unknots (not necessarily circles.)

Let C be a closed vortex line, oriented so that arc length s increases in the direction of the vorticity vector. Then

$$\Omega \equiv \oint_C \omega \cdot \mathbf{ds} = \oint_C |\omega| ds > 0. \tag{90}$$

Lemma 1 *Let ω be the vorticity of an Euler flow, defined for $0 < t < T$. Then for these times*

$$\frac{d\Omega}{dt} \equiv \Omega_t = 2 \oint_C \omega \cdot \mathbf{du}. \tag{91}$$

Proof: Since a vortex line is a material invariant of the flow, a point on the line is a Lagrangian variable $\mathbf{x}(\mathbf{a}, t)$. If the Jacobian of the Lagrangian map $\mathbf{a} \to \mathbf{x}$ is $\{J_{ij}\}$, we have

$$\begin{aligned} \frac{d\Omega}{dt} &= \oint_C \frac{D\omega_i}{Dt} dx_i + \oint_C \omega_i \frac{DJ_{ij}}{Dt} da_j \\ &= \oint_C \omega_j \frac{\partial u_i}{\partial x_j} dx_i + \oint_C \omega_i \frac{\partial u_i}{\partial x_k} J_{kj} da_j. \end{aligned} \tag{92}$$

But on C we have $dx_i = \omega^{-1}\omega_i ds$ and $\omega_j \frac{\partial u_i}{\partial x_j} = \omega \frac{\partial u_i}{\partial s}$, so that

$$\frac{d\Omega}{dt} = \oint_C \omega_i \frac{\partial u_i}{\partial s} ds + \oint_C \omega_i du_i = 2 \oint_C \omega \cdot \mathbf{du}, \tag{93}$$

which completes the proof.

Lemma 2

$$\frac{d\Omega_t}{dt} = \frac{d^2\Omega}{dt^2} = 2 \oint_C \omega |\frac{\partial \mathbf{u}}{\partial s}|^2 ds + 2 \oint_C \frac{\partial \omega}{\partial s} \cdot \nabla p ds. \tag{94}$$

Thus the second derivative brings the pressure into the equation. Proof: We need to compute D/Dt of $2\omega_i ds_j \frac{\partial u_i}{\partial x_j}$. We have

$$\frac{D\omega_i}{Dt} = \omega_k \frac{\partial u_i}{\partial x_k}, \frac{Dds_j}{Dt} = ds_k \frac{\partial u_j}{\partial x_k}, \tag{95}$$

and

$$\frac{D\frac{\partial u_i}{\partial x_j}}{dt} = -\frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{\partial u_k}{\partial x_j} \frac{\partial u_i}{\partial x_k}. \tag{96}$$

Using these results we see that two terms cancel and that

$$\oint_C \omega_i ds_j \frac{\partial^2 p}{\partial x_i \partial x_j} = -\oint_C \frac{\partial \omega_i}{\partial s} \frac{\partial p}{\partial x_i} ds, \tag{97}$$

we obtain the stated result.

We note that the result of Lemma 1 may be written

$$\frac{d\Omega}{dt} = 2\oint_C \omega_i ds_j \frac{\partial u_i}{\partial x_j} = 2\oint_C \omega_i \omega_j \omega^{-1} \frac{\partial u_i}{\partial x_j} ds \tag{98}$$

and therefore as

$$\frac{d\Omega}{dt} = \oint_C \omega_i \omega_j \omega^{-1} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) ds. \tag{99}$$

Now the Biot-Savart law giveus us the velocity in terms of vorticity,

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{\mathbf{y} \times \omega'}{y^3} dV', \quad , \mathbf{y} = \mathbf{x} - \mathbf{x}'. \tag{100}$$

Differentiating this expression as a distribution we obtain the following preincipal value result for the rate of strain tensosr:

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = \frac{3}{4\pi} PV \int y^{-5} [\mathbf{y} \times \omega' \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{y} \times \omega'] dV'. \tag{101}$$

Letting $\omega_i = \omega t_i$, we may write a complete integral for $d\Omega/dt$ as follows:

$$\frac{d\Omega}{dt} = \frac{3}{2\pi} \oint_C PV \int \omega \omega' y^{-5} \mathbf{t} \cdot \mathbf{y} (\mathbf{t}' \times \mathbf{t}) \cdot \mathbf{y} dV' ds \tag{102}$$

5.7 Interacting vortex tubes

Suppose now that the curve C is the centerline of a vortex tube of infinitesimal cross-section but unit flux. The length of the tube is given by

$$L = \oint_C \mathbf{t} \cdot \mathbf{ds}. \tag{103}$$

Thus

$$\frac{dL}{dt} = \oint_C \mathbf{t} \cdot d\mathbf{u}, \tag{104}$$

since $d\mathbf{t}$ is orthogonal to $\mathbf{t}$. Thus

$$\frac{dL}{dt} = \frac{1}{2} \oint_C t_i t_j (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) ds. \tag{105}$$

If $\mathbf{u}$ is caused by a second closed unit vortex tube with centerline C', we get an equation for the rate of stretching of C in the form

$$\frac{dL}{dt} = \frac{3}{4\pi} \oint_C \oint_{C'} y^{-5} \mathbf{t} \cdot \mathbf{y} (\mathbf{t}' \times \mathbf{t}) \cdot \mathbf{y} ds' ds. \tag{106}$$

Note that we may integrate the right hand side of (106) by parts with respect to s to obtain a simpler form,

$$\frac{dL}{dt} = \frac{1}{4\pi} \oint_C \oint_{C'} y^{-3} \mathbf{y} \cdot (\mathbf{t}' \times \mathbf{t}_s) ds' ds. \tag{107}$$

Combining two such expressions (106)we obtain

$$\frac{dL}{dt} + \frac{dL'}{dt} = \frac{3}{4\pi} \oint_C \oint_{C'} y^{-5} (\mathbf{t} + \mathbf{t}') \cdot \mathbf{y} (\mathbf{t}' \times \mathbf{t}) \cdot \mathbf{y} ds' ds. \tag{108}$$

This follows from interchanging primed and unprimed quantities, including the differentiations of $\mathbf{u}$.

To give an example consider the two vortex rings of radius R in $x - y$ plance, centered at the common z-axis on the planes $z = 0, H$. The lower curve is oriented counter-clockwise from above, the upper curve clockwise. Then $\mathbf{y} = [R(\cos\theta - \cos\theta'), R(\sin\theta - \sin\theta'), H]$ and $\mathbf{t} = (\sin\theta, -\cos\theta), \mathbf{t}' = (-\sin\theta', \cos\theta')$, and so we get an equation for the rate of stretching of C in the form

$$\frac{dR}{dt} = \frac{RH}{4\pi} \int_0^{2\pi} \frac{cos\psi}{[2R^2(\cos\psi + 1) + H^2]^{3/2}} d\psi, \quad \psi = \theta - \theta', \tag{109}$$

We remark that we also have, from local self-induction, that dH/dt is proportional to $1/R$. This lead to rings which expand with a radius proportional to $\sqrt{t}$ for large time. This describes the interaction of two rings which approach each other and expand their radii. Ultimately the line description breaks down, of course, as the two rings get close together.

5.8 Axisymmetric flow reconsidered

We reconsider axisymmetric flow without swirl and redo the proof of global regularity given in (5.5) within the present framework. We suppose that the initial vortical field is

$$(\omega_z, \omega_r, \omega_\theta) = (0, 0, \omega_0(\zeta, \rho)). \tag{110}$$

Thus the vorticity field consists of circular unknots. Lagrangian variables for the Euler flow then have the form $\mathbf{x} = \mathbf{x}(t; \zeta, \rho, \theta)$. We let$\zeta, \rho$ range over the closed compact set, D. The vorticity field for subsequent times is given by

$$\omega(\mathbf{x}(t; \zeta, \rho, \phi), t) = \rho^{-1} \omega_0(\zeta, \rho) \frac{\partial \mathbf{x}}{\partial \phi}, \quad (\zeta, \rho) \in D, 0 \leq \phi \leq 2\pi. \tag{111}$$

That is, $\zeta, \rho, \phi)$ are Lagrangian variables. We want now to consider the effect of all vorticity on the stretching of a single unknot C of instantaneous length $L = 2\pi R(t)$. Using (107)

$$2\pi \frac{dR}{dt} = \frac{1}{4\pi} \oint \int_{R^3} \Big[|\mathbf{y}|^{-3} \mathbf{y} \cdot (\mathbf{t}' \times \mathbf{t}_s) \frac{\partial s'}{\partial \phi} \frac{\omega_0}{\rho} (\zeta, \rho) \rho d\rho d\zeta d\phi \Big] ds. \tag{112}$$

Let V_0 denote the support of the vorticity in Lagrangian space, as well as the volume of this support. We assume that $\max_{V_0} \frac{|\omega_0|}{\rho} \leq C$. Then we may estimate (112) in a manner similar to what was done in section (5.5). For some $R_0 > 0$we divide the inner integral into to parts, $I_i, i = 1, 2$,

$$I_1 = \int_{|\mathbf{y}| \leq R_0} \Big[\cdot \Big], \quad I_2 = \int_{|\mathbf{y}| \geq R_0} \Big[\cdot \Big]. \tag{113}$$

We see that , since we may take $d\theta' = d\phi$, and see that $r' \leq R + R_0$, and $\mathbf{t}_s = R^{-1}$, we have

$$|I_1| \leq C\frac{R+R_0}{R}(4\pi R_0). \tag{114}$$

Also

$$|I_2| \leq C\frac{1}{R_0^2 R}(R+R_0)V_0, \tag{115}$$

where we have used $\omega_0(\zeta,\rho)/\rho = \omega(z,r)/r$. Combining these results,

$$2\pi|\frac{dR}{dt}| \leq \frac{1}{2}R\big[C\frac{R+R_0}{R}(4\pi R_0) + C\frac{1}{R_0^2 R}(R+R_0)V_0\big]. \tag{116}$$

Setting $4\pi R_0^3 = V_0$ we obtain

$$|\frac{dR}{dt}| \leq 2CR_0(R+R_0), \tag{117}$$

and so $R \leq (R_0 + R(0))e^{2CR_0 t} - R_0$.

Conjecture: *The maximum vorticity of an axisymmetric Euler flow without swirl in R^3 grows at a rate which can be bounded above by Ct^α for some positive number α.* That is, the exponential bound we have established does not adequately account for the constraints of topology for this class of flows. We shall indicate below a proof that the actual bound should be a multiple of t^2.

6 Geometry of Euler flows- variational methods

6.1 Hamilton's Principle and Hamiltonian structure

The canonical Hamiltonian representation of dynamics takes the form (see the paper by Shepherd in [3])

$$\frac{\partial \mathbf{v}}{\partial t} = J\frac{\partial \mathcal{H}}{\partial \mathbf{v}}, \tag{118}$$

where $\mathbf{v} = (q_1, \ldots, q_n, p_1, \ldots, p_n)^T$ are position and momentum field variables,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{119}$$

where I is the $n \times n$ identity matrix, and $\mathcal{H}$ is the Hamiltonian function.

Hamilton's Principle states that the evolution of the dynamical system must be such that $\delta \int \mathcal{L} dt = 0$ for some Lagrangian and all variations of the coordinates (vanishing at the endpoints of the time interval of integration $\mathbf{x}(t)$ involved in the definition of $\mathcal{L}$. In the present context we take

$$\mathcal{L} = \int_D \frac{1}{2}\Big[\frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \mathbf{x}}{\partial t}\Big] + p(\mathrm{Det}(\mathbf{J}) - 1)dV. \tag{120}$$

Here the time derivatives are at fixed particle label, and p is a Lagrange multiplier depending upon particle and time.[3] Since $\delta\mathrm{Det}(\mathbf{J}) = \nabla\cdot\delta\mathbf{x}$ we obtain

$$\delta\int\mathcal{L}dt = \int\int_D -[\frac{\partial^2\mathbf{x}}{\partial t^2} + \nabla p]\cdot\delta\mathbf{x}dVdt = 0, \tag{121}$$

Thus the Euler equations are obtained as the Euler-Lagrange equations under the variation, plus $\mathrm{Det}(\mathbf{J}) = 1$ as a condition on the Lagrangian coordinates.

In fluid mechanics, the Hamiltonian formulation arises very simply in 2D kinematics. the streamfunction $\psi(x, y, t)$ of a 2D incompressible velocity field satisfies

$$\frac{\partial x}{\partial t} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial y}{\partial t} = -\frac{\partial\psi}{\partial x}, \tag{122}$$

where x, y are Lagrangian coordinates.

In the case of the 3D Euler equations, we define a Hamiltonian analogously to what is done in finite systems, as

$$\mathcal{H} = \int_D \big[\mathbf{u}\cdot\frac{\partial\mathbf{x}}{\partial t} - \mathcal{L}\big]dV = \int_D \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - p(\mathrm{Det}(\mathbf{J}) - 1)dV. \tag{123}$$

Note that variations in $\mathbf{u}$ and $\mathbf{x}$ are taken independently, and we make use of the variational derivative. Let $F = \int_D f(\mathbf{x})dV$. Then $\delta F/\delta\mathbf{x}$ is defined by $\delta F \int_D]\delta F/\delta\mathbf{x}]\cdot\delta\mathbf{x}dV$ for all variations $\delta\mathbf{x}$.

We thus see from (123) that

$$\mathbf{u} = \frac{\partial\mathbf{x}}{\partial t} = \frac{\delta\mathcal{H}}{\delta\mathbf{u}}, \quad \nabla p = -\frac{\partial\mathbf{u}}{\partial t} = \frac{\delta\mathcal{H}}{\delta\mathbf{x}}. \tag{124}$$

This is a Hamiltonian system in "canonical" form, but with reference to an infinite dimensional system.

In fact, the space $\mathcal{D}$ is so complicated that this formally elegant result is not very useful in practice. We indicate below some of this complexity.

A better approach to a Hamiltonian theory of the 3D Euler flow problem is to focus on the vorticity field in place of the vector of coordinates and momenta, and not seek a canonical structure. We note first that if we define the Hamiltonian as simply the total kinetic energy,

$$\mathcal{H} = \int_D \frac{1}{2}|\mathbf{u}|^2 dV, \tag{125}$$

and set $\mathbf{u} = \nabla\times\mathbf{A}$, then we have

$$\frac{\delta\mathcal{H}}{\delta\omega} = -\mathbf{A}. \tag{126}$$

Problem 8 *Verify* (126).

[3] Do not confuse the J symbol, used here for the Jacobian, with the J operator in Hamiltonian theory.

Thus the vorticity equation may be written as

$$\frac{\partial\omega}{\partial t} = J\frac{\delta\mathcal{H}}{\delta\omega}, \tag{127}$$

where J denotes the operator

$$J = -\nabla\times(\omega\times\nabla\times(\cdot)). \tag{128}$$

To qualify as a Hamiltonian representation, J must determine a bracket

$$[f,g] = (\frac{\delta f}{\delta\mathbf{u}}, J(\frac{\delta g}{\delta\mathbf{u}}), \tag{129}$$

which has the right properties: it must be bilinear, skew-symmetric, and satisfy the Jacobi identity. These hold for (128), so that (128) is a Hamiltonian representation of the vorticity equation.

It is however not canonical, because $J\mathbf{A} = 0$ in general has non-trivial solutions, namely the possible steady Euler flows. A functional $\mathcal{C}$ of ω having the property that

$$[\mathcal{C},\mathcal{G}] = (\frac{\delta\mathcal{C}}{\delta\omega}, J\frac{\delta\mathcal{G}}{\delta\omega}) = 0, \quad \text{for all } \mathcal{G} \tag{130}$$

is said to be a *Casimir*. One important Casimir for Euler is the total helicity ,

$$\mathcal{C} = \int_D \mathbf{u}\cdot\omega dV. \tag{131}$$

(We have previously used $\mathcal{H}$ for helicity, but this symbol has been adopted locally for the Hamiltonian.) We have $\frac{\delta\mathcal{C}}{\delta\omega} = 2\mathbf{u}$. From the extremal property for steady flows (variation of $\mathbf{u}$ be transport under a flow), we would thus expect that *for any steady Euler flow, there must exist a Casimir* $\mathcal{C}$ *such that*

$$\frac{\delta\mathcal{C}}{\delta\omega} = \frac{\delta\mathcal{H}}{\delta\omega} = -\mathbf{A}. \tag{132}$$

In the case of helicity, these are the Beltrami flows with constant proportionality, a very small subset of steady Euler flows. However other methods may be used to identify all Casimir invariants and show that all Euler steady flows may be characterized as extremals with respect to variations of ω, see [17]

6.2 Unsteady Euler flows as geodesics

We have observed above that the action

$$\mathcal{A}\{\mathbf{x}\}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_D \frac{1}{2}|\frac{\partial\mathbf{x}}{\partial t}|^2 dV dt \tag{133}$$

is stationary under volume preserving variations $\delta\mathbf{x}$, provided that these variations vanish at $t = t_1, t_2$ and the flow field as a whole is an Euler flow. This means that in the space $\mathcal{D}$, the fluid flows are geodesics in the topology of the energy, a point that Arnold emphasized in his early papers.

Shnirelman (see his article in [2]), has explored the geometry of $\mathcal{D}$ from this viewpoint, and has found some unexpected properties which help to understand how complicated the space $\mathcal{D}$ actually is.

Let us denote by $\mathbf{g}$ a certain element of $\mathcal{D}$, that is, a diffeomorphism of D into itself, generated by some flow with time as a parameter. For $\mathbf{g}$ we define both the action, as above, and the *length*

$$L\{\mathbf{g}\}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \Big(\int_D |\frac{\partial \mathbf{x}}{\partial t}|^2 dV\Big)^{1/2} dt \tag{134}$$

Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be two elements of $\mathcal{D}$. We define the *distance between the maps*, by

$$dist(\mathbf{g}_1, \mathbf{g}_2) = inf\ L\{\mathbf{h}\}\Big|_0^1, \tag{135}$$

where the infimum is taken over all smooth flows $\mathbf{h}(\mathbf{x}, t)$ such that $\mathbf{h}(\mathbf{x}, 0) = \mathbf{g}_1(\mathbf{x})$ and $\mathbf{h}(\mathbf{x}, 1) = \mathbf{g}_2(\mathbf{x})$. We say that a map $\mathbf{g}$ is *attainable* if $dist(\mathbf{e}, \mathbf{g}) < \infty$, where $\mathbf{e}$ denotes the identity map. In other words, a map is attainable if there exists a smooth flow of finite length (not necessarily an Euler flow) whose Lagrangian coordinates yield the map at time 1. Finally, we define the *diameter* of $\mathcal{D}$ as

$$diam(\mathcal{D}) = \sup_{\mathbf{g},\mathbf{h}\in\mathcal{D}} dist(\mathbf{g}, \mathbf{h}). \tag{136}$$

We then have the following result;

Theorem 4 *Let d be the unit cube. (1) In two dimensions, $diam(\mathcal{D}) = \infty$. (2) In three dimensions, $diam(\mathcal{D}) < \infty$.*

The idea of the proof in 2D (I have not found a complete account, so must guess) is to work inside a circle instead of a square (an inessential difference), and to consider a "twist map"

$$(r, \theta) \to (r, \theta + N\psi(r)) = \mathbf{g}_N(r, \theta), \tag{137}$$

It then is shown that $dist(\mathbf{e}, \mathbf{g}_N) \geq cN$ and so grows indefinitely with N. Of course for large N points move around the circle many times and so it would seem possible to make angles *mod* 2π and keep the distance bounded. But remember we must use smooth diffeomorphisms. Suppose that a particle moves $2\pi + \epsilon$ while a nearby one moves $2\pi - \epsilon$. We see that a discontinuity in the map could be generated by this procedure. In three (or higher) dimensions the important point is that the extra dimensions enable shorter returns to be found, so that one can always keep the distance under control. Shnirelman proves the result by an ingenious construction involving the division of the cube into small sub-cubes. Within each sub-cube, the map is smooth, and the global map is constructed by continuous permutations of the sub-cubes. He is then able to show that the he can find a discrete map of this kind, which is arbitrarily close to a smooth map, which makes the distance finite.

Shnirelman has also proved an interesting result concerning Euler flows as geodesics. We say that a flow $\mathbf{u}$ "connects" $\mathbf{e}$ to an element $\mathbf{g} \in P\mathcal{D}$ if the Lagrangian coordinates of $\mathbf{u}$ yield $\mathbf{g}$ at time 1.

Theorem 5 *In three dimensions, there exists an element $\mathbf{g} \in \mathcal{D}$, such that for an smooth flow $\mathbf{u}$connecting $\mathbf{e}$ and $\mathbf{g}$, there is another smooth flow $\tilde{\mathbf{u}}$ connecting $\mathbf{e}$ and $\mathbf{g}$, such that the action is smaller, $\mathcal{A}\{\tilde{\mathbf{u}}\}\big|_0^1 < \mathcal{A}\{\mathbf{u}\}\big|_0^1$. Thus the minimal action cannot be achieved.*

This result is analogous to non-existence of minimizers in certain classical variational problems, e.g. the shortest smooth curve connecting points A and B in R^2, such that the tangents at the points are orthogonal to the line joining the two points. It points to the complexity of the objects in $\mathcal{D}$, and the inadequacy of the present Euler flow theory.

The proof of the theorem utilizes a $\mathbf{g}$ of the form $(\mathbf{h}(x, y), z)$, i.e. a 2D map in planes $z =$ constant. The proof is technically involved because of its use of a kind of generalized Euler flow defined probabilistically. However it relies on the profound difference in the geometry of two and three space dimensions.

To sum up, in three dimensions we know that all fluid states are accessible by elements of $\mathcal{D}$ of finite length. We also know that in a formal sense an Euler flow connecting two fluid states will minimize the action and so be a geodesic in $\mathcal{D}$. However this minimization problem need not have a classical solution.

6.3 Braids

This term may be used in several ways in the discussion of solutions of Euler's equations. First take $D = D_2$ as two-dimensional and consider the cylinder $D_2 \times \{t|0 \leq t \leq 1\}$. The orbits of a fluid particle moving over the time interval $[0, 1]$ are now strands connecting the two ends of the cylinder. The totality of these strands corresponding to a flow in D_2 form a *continuous braid*. Euler flow in 2D can thus be represented as braids in R^3. Alternatively, consider *steady* vorticity fields in $D_2 \times \{z|0 \leq z \leq 1\}$. Any vortex line connects the two ends of the cylinder in a 1-1 fashion. Does there exist a steady Euler flow with this structure?

This last braid construction has a widely known analog in magnetostatics, which is actually more natural as a physical problem. In fact this form of the problem has been proposed by Parker as a model for coronal heating of the sun. Here the strands are magnetic field lines, and the question is what happens as the field lines relax to a steady configuration. Recall that this magnetostatic equilibrium is mathematically (apart from boundary conditions) equivalent to steady Euler flow when $\mathbf{u}$ replaces $\mathbf{B}$.

The nice thing about a braid is that it offers a fairly clean way to describe a knot, but also has the above applications. In Figure 6 we show a (left) trefoil knot, and how it leads naturally to a braid. We can then develop a symbolic algebra of braids by counting the strands from left to right at the top, and as we descend the braid apply from the left a factor σ_i if the ith strand crosses over the i+1st strand, and the factor σ_i^{-1} if it crosses under. Thus from Figure 6(d) we see that the left trefoil knot has the symbol $\sigma_1\sigma_1\sigma_1 = \sigma_1^3$.

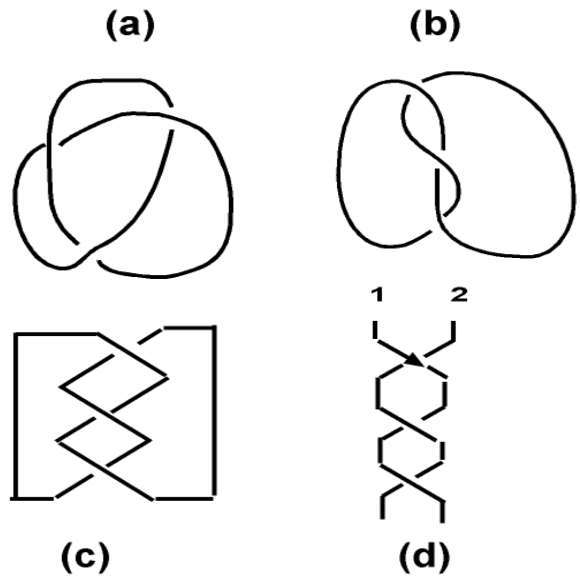

Figure 6. (a) A (left) trefoil knot. (b) An equivalent knot. (c) Regular representation. (d) Braid representation.

Of course we can do this for any finite number of strands, and in fact this representation of n-stranded braids (or their equivalence classes) forms a group.

6.4 The Parker problem

Out interest here is continuous braids and the Parker problem mentioned above is a concrete application of this idea. We consider a representation of a steady magnetic field in terms of the *Euler potentials* α, β,

$$\mathbf{B} = \nabla\alpha \times \nabla\beta. \tag{138}$$

Let a magnetic field be given in a *stretched* cylindrical domain $D_2 \times \{\zeta | 0 \leq \zeta \leq 1\}$, where we set $\zeta = \epsilon z$ and we regard the diameter of D_2 as of order unity. We have seen that force-free magnetic fields are analogous to Beltrami velocity fields, and that these fields may be obtained by minimization of energy subject to fixed helicity.

Another way that a force-free field can be obtained in the present geometry is by prescribing the normal component of the magnetic field on the ends of the cylinder, see [18]. We thus seek, in the stretched geometry, potentials having the expansions

$$\alpha = A(x, y, \zeta) + \epsilon^2 C(x, y, \zeta) + O(\epsilon^4), \tag{139}$$

$$\beta = B(x, y, \zeta) + \epsilon^2 D(x, y, \zeta) + O(\epsilon^4). \tag{140}$$

We require that $\mathbf{B} \cdot \mathbf{i}_z = 1 + O(\epsilon^2)$, so that necessarily

$$A_x B_y - A_y B_x \equiv \partial(A, B)/\partial(x, y) = 1. \tag{141}$$

One finds from these relations, writing $\mathbf{B} = (B_x, B_y, B_z)$,

$$B_x = \epsilon \partial(A, B)/\partial(y, \zeta), \quad B_y = \epsilon \partial(A, B)/\partial(\zeta, x), \tag{142}$$

$$B_z = 1 + \epsilon^2 \big[\partial(A, D)/\partial(x, y) - \partial(B, C)/\partial(x, y)\big]. \tag{143}$$

Note that this is a stretched braid which does allow non-trivial knots, since the $O(\epsilon)$ transverse fields can intertwine over a distance $z = O(\epsilon^{-1})$.

The minimization of energy will now be carried out subject to(141). Introducing a multiplier $\lambda(x, y, \zeta)$, we have

$$\delta((M + N + 2K) = 0, \tag{144}$$

where

$$M = \int_D (\partial(A, B)/\partial(y, \zeta))^2 + (\partial(A, B)/\partial(x, \zeta))^2 dV, \tag{145}$$

$$N = \int_D 2\big(\partial(A, D)/\partial(x, y) - \partial(B, C)/\partial(x, y)\big) dV, \tag{146}$$

$$K = \int_D \lambda \partial(A, B)/\partial(x, y) dV. \tag{147}$$

Performing now the first variation, we assume that $\delta A = \delta B = 0$ at the ends of the cylinder (magnetic lines tied there), and discard all lateral surface integrals by necessary boundary conditions there or else considering the walls to be at infinity with compact support for the magnetic field. Wed then obtain the following Euler-Lagrange equations:

$$\partial(A, \partial(A, B)/\partial(y, \zeta))/\partial(y, \zeta) + \partial(A, \partial(A, B)/\partial(x, \zeta))/\partial(x, \zeta)$$
$$+ \partial(A, \lambda)\partial(x, y) = 0, \tag{148}$$

$$\partial(B, \partial(A, B)/\partial(y, \zeta))/\partial(y, \zeta) + \partial(B, \partial(A, B)/\partial(x, \zeta))/\partial(x, \zeta)$$
$$+ \partial(B, \lambda)\partial(x, y) = 0, \tag{149}$$

$$\partial(A, B)\partial(x, y) = 1. \tag{150}$$

While these expressions look formidable, they actually have a simple interpretations in terms of the magnetic field structure or alternatively an Euler flow.

If we multiply (148) by $B_x(B_y)$ and (149) by $A_x(A_y)$ and subtract, we obtain two equations have the form

$$U_\zeta - V(V_x - U_y) + P_x = 0, \quad V_\zeta + U(V_x - U_y) + P_y = 0, \tag{151}$$

$$P = -\lambda, U = \partial(A, B)/\partial(y, \zeta), V = -\partial(A, B)/\partial(x, \zeta), \tag{152}$$

where $(U, V) = (B_x, B_y)$: (Note that $U_x + V_y = 0$ by (150).) The notation here is meant to reflect that fact that these equations are those of a 2D incompressible fluid with ζ playing the role of time.

But first note the following:

Theorem 6 *If the Euler potentials are close to Cartesian coordinates, i.e.* $A = x + \delta A, B = y + \delta B$, *where the variations either vanishes at lateral infinity, or else* $(\delta A_{\zeta\zeta}, \delta B_{\zeta\zeta}) \cdot \mathbf{N} = 0$, *then we have*

$$\delta A \approx \Psi_y, \delta B \approx -\Psi_x, \quad \Psi = \Psi_0(x, y) + \zeta(\Psi_1(x, y) - \Psi_0(x, y)). \tag{153}$$

To prove this, make the substitutions $A = x + \delta A, B = y + \delta B$ in M, N, K and take first variations. This is easily seen to yield

$$\delta A_x + \delta B_y \approx 0, \delta A_{y\zeta\zeta} - \delta B_{x\zeta\zeta} \approx 0. \tag{154}$$

Introducing the streamfunction Ψ as in (153), we have $\Psi_{xx\zeta\zeta} + \Psi_{yy\zeta\zeta} = 0$. Under the stated conditions we then must have $\Psi_{\zeta\zeta}$ is independent of x, y, and we may take it as zero since only $\delta A, \delta B$ are of interest. Thus ψ is linear in ζ and we are done. Thus if $\Psi_0(x, y)$ and $\Psi_1(x, y)$ are smooth functions, then so is $\mathbf{B}$. If these two functions have singularities, the a "ribbon" singularity extends through the cylinder.

But our analogy with a 2D flow allows us to make a similar conclusion regarding more general fields. Since we have global regularity for 2D Euler, the only way to have singularities in the equilibrium field is for the " initial" (U,V), at $\zeta = 0$, to possess singularities. These then are carried "forward" by the flow, yielding a ribbon which extends down the cylinder and connects the two ends. Typically the singularity in question will be a vortex sheet as in Figure 5.

What is not known is whether the "unstretched" geometry admits singularities in $\mathbf{B}$ in the interior of the cylinder even though the Euler potentials are smooth function of x, y at either end.

7 Nearly 2D Euler flows

We have seen the great difference between 2 and 3 dimensions in every aspect of Euler flow theory. Is there any theoretical approach that lies in between these two disparate cases? One can in principle consider fluid dynamics in a space of fractional dimension, but I refer here to a geometrical approximation analogous to the stretched Parker problem. If one similarly "stretches" vortical structures in R^3 so that they are quasi-2D, what can be said about the subsequent evolution of the stretched configuration? The disappointing fact seems to be that all that can be said is that in a finite time, in general, whatever approximations were used to obtain quasi-two-dimensionality will cease to be valid, and so the structure must revert to one which is essentially three-dimensional. If this is to be taken as fact, any attempt to find some restricted class of "simpler" Euler flows in which a global regularity result might be proved is doomed to failure, apart of a trivial group of Euler flows which are locally stable when subjected to small disturbances.

An class of stretched Euler flows indicating the above assertion has been motivated by a "stretched Taylor-Green problem" [15]. Taylor and Green consider the IVP for 3D Euler for initial conditions of a form typified by $\mathbf{u}_0 = (\psi_y, \psi_x, 0)$ with $\psi = \sin x \sin y \sin z$. The flow can be described initially as a periodic array of vortical cells with axes aligned with the z-axis, such that in each cell the axial vorticity is periodic in z with period 2π. Since the circulation carried by a cell is also periodic, vorticity threads through the boundary of the cells. As time progresses this simple alignment becomes very distorted, and Taylor and Green were interested in the creation of small eddies from larger ones by this nonlinear process.

A 'stretched" version of this problem would take $\psi = \sin x \sin y \sin \zeta$ where $\zeta = \epsilon z$. More generally, we might assume that in a general cylindrical region $D_2 \times R^1$ we have an initial velocity as above but with $\psi = \psi_2(x, y) \sin \zeta$ where ψ_2 is topologically a set of closed nested streamlines with the same orientation. We will summarize how this restricted IVP can be analyzed, and indicate how it leads to a conclusion that the stretched configuration breaks down in finite time.

We write

$$\mathbf{u}(x, y, z, t; \epsilon) = \mathbf{q}(x, y, \zeta, \tau; \epsilon) + w(x, y, \zeta, \tau; \epsilon)\mathbf{k}. \tag{155}$$

Here $\tau = \epsilon t$. The then expand in ϵ:

$$(\mathbf{q}, w, p) = (\mathbf{q}_0, w_0, p_0) + \epsilon(\mathbf{q}_1, w_1, p_1) + O(\epsilon^2). \tag{156}$$

It is easy to see that, once these expansions are inserted into the 3D Euler equations and the $O(1)$ terms taken, we find that $(\mathbf{q}_0, w_0, p_0)$ satisfies the 2D steady 2D Euler system augmented by $\mathbf{q}_0 \cdot \nabla w_0 = 0$. Gathering the terms of $O(\epsilon)$ we obtain, dropping the subscripts "0",

$$\mathbf{q} \cdot \nabla w_1 = -Dw + q^2/2 - H_\zeta, \tag{157}$$

$$\mathbf{q} \cdot \nabla \omega_1 = -D\omega + \omega w_\zeta - (\mathbf{q}_\zeta \times \mathbf{k}) \cdot \nabla w, \tag{158}$$

$$\nabla \cdot \mathbf{q}_1 = -w_\zeta. \tag{159}$$

We have defined

$$D = \frac{\partial}{\partial \tau} + w\frac{\partial}{\partial \zeta} + \mathbf{q}_1 \cdot \nabla. \tag{160}$$

Since we have assumed a local topology of nested closed contours, a compatibility condition must be satisfied by the right-hand sides of (157) and (158). The left-hand sides vanish under the operation of contour averaging,

$$\langle\cdot\rangle \equiv \oint q^{-2}\mathbf{q}\cdot\mathbf{ds} = \oint \frac{ds}{q}. \tag{161}$$

If the contour over which the averaging takes the constant value $\psi(x, y, \zeta, \tau)$, the following properties of $\langle\cdot\rangle$ can be established:

$$\langle 1\rangle = A_\psi,\ \langle\psi_\zeta\rangle = -A_\zeta,\ \langle\psi_\tau\rangle = -A_\tau. \tag{162}$$

Here A is the area within the contour carrying this value of ψ, as a function of ζ, τ.

To prove the first of (162), note that

$$\delta A = A_\psi \delta\psi = \oint \delta n ds = \delta\psi \oint \frac{\delta n}{\delta\psi} ds = \delta\psi \oint \frac{ds}{dq} = \delta\psi\langle 1\rangle. \tag{163}$$

The other two identities can be established by noting that ψ_ζ or ψ_τ both vanish under the condition that s, ψ are being held fixed. Thus, for example $0 = \psi_\zeta|_{n,s} + \psi_n n_\zeta|_{\psi,s}$. Contour averaging then establishes the second relation, and similarly for the third. We also note that

$$\int\int (\cdot) dA = \int \langle\cdot\rangle d\psi. \tag{164}$$

These relations may be used to evaluate the compatibility relations obtained from (157),(158), and we omit the details, which mainly involve careful use of the chain rule.

The final results utilize the contours $\psi=$ constant for each ζ, τ as "radial" coordinates. The cross-contour velocity U is defined by

$$U \equiv \langle D\psi\rangle / A_\psi. \tag{165}$$

We also define

$$D_\psi = \partial_\tau + w\partial_\zeta + U\partial_\psi. \tag{166}$$

We note that $\langle D(\cdot)\rangle = A_\psi D_\psi(\cdot)$, where $(\cdot)$ can be any function of ψ, ζ, τ. It may be shown that

$$D_\psi A_\psi + A_\psi(w_\zeta + U_\psi) = 0. \tag{167}$$

The other equations resulting from contour averaging are then

$$A_\psi(D_\psi w + H_\zeta) = \Gamma_\zeta, \tag{168}$$

$$\Gamma = \int^\psi H_\psi A_\psi d\psi. \tag{169}$$

$$D_\psi \Gamma = 0. \tag{170}$$

Note that Γ is the circulation around a contour, so that (170) is just Kelvin's theorem, while (168) is the conservation equation for ζ-momentum. We now have equations (167)–(170) for the five unknowns A, w, U, H, Γ, and so lack one relation. This last relation must relate the area function $A(\psi)$ to the local set of contours. This must be obtained by solving $\nabla^2\psi = H_\psi(\psi)$ at fixed values of ζ, τ. The solution in the cylinder of this elliptic problem gives $A(\psi)$. This kind of PDE system was studied by Harold Grad in connection with plasma models of a similar kind, and has been referred to as a *generalized partial differential equation*. The distinguishing feature of a GPDE is that on of the "independent" variables, in this case ψ, depends upon the solutions to the entire system (since we must know H to solve for ψ). The solution of a GPDE thus involves iterations between the trial area function A and it's resulting H, and the next iterate of A determined by the solution to the local elliptic problem.

A further reduction of the problem is possible, to a form that i effectively in axysymmetric variables, a so–called "Schwarz symmetrization". (This is related to the reduction to action-angle variables in classical dynamics.) We define an effective cylindrical radius by

$$r_e^2 = A/\pi. \tag{171}$$

The effective swirl component of velocity is then

$$v_e = \frac{\partial\psi}{\partial r_e}. \tag{172}$$

Finally the effective radial velocity u_e is defined by

$$D_\psi = D_e = \frac{\partial}{\partial\tau} + w\frac{\partial}{\partial\zeta} + u_e\frac{\partial}{\partial r_e}. \tag{173}$$

We then obtain the following effective axisymmetric system:

$$\frac{\partial w}{\partial\zeta} + r_e^{-1}\frac{\partial r_e v_e}{\partial r_e} = 0, \tag{174}$$

$$2\pi r_e\big(D_e w + H_\zeta\big) - v_e\Gamma_\zeta = 0, \tag{175}$$

$$2\pi H_{r_e} = v_e\Gamma_{r_e}, \quad D_e\Gamma = 0. \tag{176}$$

Note that, if $\Gamma = 2\pi r_e v_e$ and $H = p + v_e^2/2$, as for true axisymmetric (stretched) flow with swirl, then the system (174)–(176) reduces to

$$D_e w + p_\zeta = 0, p_{r_e} = v_e^2/r_e, D_e(r_e v_e) = 0. \tag{177}$$

This is precisely the stretched form of axisymmteric Euler flow with swirl.

This form of the problem retains the complexity of a GPDE and so further approximations can be made to facilitate numerical solutions. The simplest is to consider the stretched problem in a thin cylindrical annulus. We have mentioned above that this problem is then mathematically equivalent to the Boussinesq of a flow with density variations. The problem analogous to the Taylor-Green initial condition can be studied under these assumptions, and it is found that ζ derivatives of flow quantities become infinite in finite time, [15]. The mechanism is fairly explicit, and is illustrated in Figure 7. On either side of a section of maximum or minimum circulation, centrifugal forces at the wall, and hence the local pressure there, diminish with axial distance.Thus there is axial motion along the wall toward the reduced pressure, and this causes the vortex lines which terminate on the wall to be brought toward the section of zero circulation. This process results in finite time in infinite ζ- derivatives.

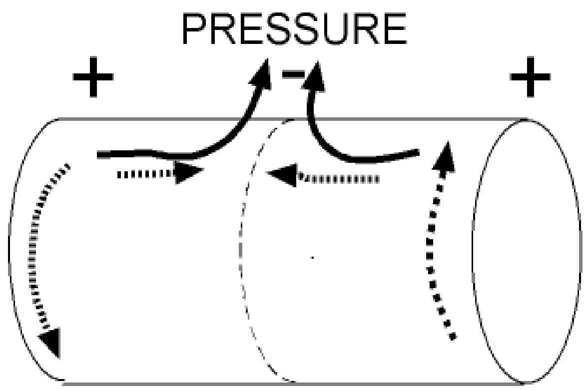

Figure 7. Blow-up in the stretched Taylor-Green type IVP in a cylinder. The generation of axial flow brings vortex lines to the sections of zero circulation. The vortex lines (solid) actually end at the boundary. The vortical circulation (curved dash lines) are shown. The straight dashed lines indicate the axial flow driven by centrifugal pressure

The problem we have considered is not the only stretched geometry that one can envision. We might also consider a distortion of vortex lines in the Parker geometry but with no penetration of vortex lines to the boundary. In other words, the circulation on each section of the cylinder is the same constant for all time. The problem here is that it appears that the expansions in ϵ must be carried out through order ϵ^2. This is an interesting case to study, but the calculations will be more elaborate.

8 Global regularity continued

8.1 Axisymmetric flow without swirl yet again

Our purpose in this section is to indicate how a close attention to vortex geometry can improve estimates of maximum vorticity and hence have an impact on issues of global regularity. We return to the problem axisymmetric flow without swirl and the conjecture that a better bound than the expoential (in time) should be possible.

Assuming that the initial vorticity satisfies $|\omega_\theta(r, z, 0)/r| < C$, it follows from the Biot-Savart law that

$$|u_r|(r,z,t) \leq C \int_{V_0} r|z-\zeta| \Big[\int_{-\pi}^{+\pi} ((r-\rho)^2 + 2r\rho(1-\cos\psi) + (z-\zeta)^2)^{-3/2} d\psi \Big] \rho d\rho d\zeta. \tag{178}$$

Here V_0 is the volume of the initial support of vorticity, assumed finite. Now $1-\cos\psi \geq k^2\psi^2, k = \sqrt{2}/\pi$, and so we may make this substitution, carry out the integral, and obtain

$$|u_r| = |\frac{dr}{dt}| \leq 2C \int_{V_0} r|\zeta|((r-\rho)^2+\zeta^2)^{-1}((r+\rho)^2+\zeta^2)^{-1/2} \rho d\rho d\zeta. \tag{179}$$

Here we have set $z=0$ since we want to arrange the vorticity to maximize the RHS and so the value of z is arbitrary. The problem then is to find an upper bound when the integral is restricted to a domain D of the ρ,ζ-plane such that $\int_D \rho d\rho d\zeta = V_0$.

Let us assume that r does not get large compared to the diameter of the initial support of vorticity. The we are done since vorticity then stays bounded for all time. If on the other hand r becomes large , then we have approximately

$$\frac{dr}{dt} \leq 2Cr \int_{V_0} |\zeta|((r-\rho)^2+\zeta^2)^{-1} d\rho d\zeta. \tag{180}$$

Note that we have taken $\rho \approx r$ since we will be maximizing the resulting integrand by concentrating the vorticity near $r=\rho, \zeta=0$. Let $\rho - r = R\cos\theta, z = R\sin\theta$. The we must maximize

$$\frac{dr}{dt} \leq 2Cr \int_0^{2\pi} \int_0^{F(\theta)} |\sin\theta| dR d\theta, \tag{181}$$

given that

$$\int_0^{2\pi} \frac{1}{2} F^2(\theta) d\theta = V_0/2\pi r. \tag{182}$$

The solution of this variational problem is

$$F = \frac{1}{\pi}\sqrt{V_0/r}|\sin\theta|. \tag{183}$$

Thus

$$\frac{dr}{dt} \leq 2C\sqrt{rV_0}. \tag{184}$$

We thus obtain a bound $r \leq C^2 V_0 t^2 + 2\sqrt{V_0 r_0} t + r_0$, or, since $|\omega_\theta| = Cr$.

$$\max|\omega_\theta| \leq C^3 V_0 t^2, t \to \infty. \tag{185}$$

This result has a simple physical interpretation in terms of advection of vortex tubes, which is typified by a thin toroidal distribution of vorticity such that the vorticity distribution of a section is as shown in Figure 8.

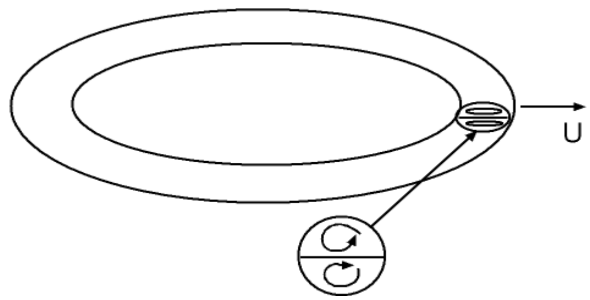

Figure 8. Expanding vortex structure yields the t^2 behavior.

The vorticity is as odd function of the dividing line, say $+\omega_0$ in the upper half and $-\omega$ in the lower. The two-dimensional vortex discussed by Batchelor (in his text, p. 535) has this structure. The important point is that, if the radius of the section is a, then $a^2 r \sim$ constant by conservation of vorticity support. But from vorticity kinematics we also have $\omega^2 \sim$ constant. Finally, the structure propagates with a velocity

$$U = \frac{dr}{dt} \sim \omega a \sim a^{-1} \sim \sqrt{r}. \tag{186}$$

If the cross-sectional structure is unchanged as the torus lengthens, then the total kinetic energy would increase, violating conservation of energy in Euler flow. Thus there must in fact be some compensating change in structure. The configuration of vorticity is similar to that of two smoke rings of opposite signed vorticity approaching each other and expanding. The compression of the two vortices onto the plane of symmetry thus is likely to be present here as well, with an accompanying effect on the rate of lengthening. We thus conjecture that the actual rate of lengthing is smaller than our estimate, although it is possible that the power t^2 is sharp. Conservation of energy would have to be imposed as a side condition examined this question.

8.2 Arbitrary unknots

The preceding calculations suggests that any properties of the vorticity that can be exploited will have an impact on estimates of vortex stretching. In the case of axisymmetric flow without swirl, a reduction on a bound from e^{ct} to t^2 seems to be accessible. Can one reduce the simple estimates which bound vorticity by $1/(t^* - t)$ to anything slower?

Two nontrivial problems with some simplifying elements suggest themselves. The first is axisymmetric flow with swirl. We have commented upon the search for blow-up in such flows–clearly the question of global regularity will be difficult to resolve even in this case, but it does deserve further work. The main consideration is that vortex lines are spirals on toroidal streamsurfaces, which may or may not be closed.

Another relatively simple topology, which we consider briefly here, is that of a continuous distribution of unknots. We may think of these flows as that class whose initial condition is an isovortical map (i.e. transport of vorticity by a smooth volume preserving flow) of an axisymmetric flow without swirl. That is, unlinked circles map into arbitrary unlinked closed loops.

We have discussed above the calculation of $dL/dt = \oint \mathbf{t} \cdot \mathbf{du}$ in the case of an unknot, where $\mathbf{t}$ is the tangent vector to the vortex line. we can write this as

$$\frac{dL}{dt} = -\oint \kappa \mathbf{n} \cdot \mathbf{u} ds, \tag{187}$$

where κ is the curvature of the vortex line and $\mathbf{n}$ the normal vector in the osculating plane. Using the Biot-Savart law we then have

$$\frac{dL}{dt} = \frac{1}{4\pi} \oint_C \oint_{S'_t} \kappa y^{-3} \mathbf{y} \cdot (\omega' \mathbf{t}' \times \mathbf{n}) dV' ds. \tag{188}$$

Here S'_t is the support of the vorticity at time t. We should point out here that it is not necessary that a vortex line acquire infinite length in order for the vorticity to become unbounded somewhere, but it iw one scenario which is worth considering separately. Prior to such a blow-up, the vortex line in question would have an arc length function which in Lagrangian notation would be $s(s_0, t)$ where s_0 is the Lagrangian coordinate. Then

$$\frac{\partial s}{\partial s_0} = \frac{\omega}{\omega_0}. \tag{189}$$

at the blow-up, $\frac{\partial s_0}{\partial s}$ must vanish at $t = t*$. Since functions prior to blowup are smooth, we have

$$\frac{\partial s_0}{\partial s} = a_2(t)(s_0 - a(t))^2 + b(t) + O((s_0 - a)^3), \tag{190}$$

where $a(t^*)$ is the Lagrangian coordinate of the blow-up point on the line, and $b(t) > 0 \to 0$ as $t \to t^*$. Of course all terms of the Taylor will in principle need to be considered as $t \to t^*$, but certain special cases lead to a segmenr of infinite length. For example if $a_2 = 1$ and we neglect all terms not exibited, integration yields the length of a small segment of the line as

$$L_\epsilon = \int_{a-\epsilon}^{a+\epsilon} \frac{ds_0}{(s_0 - a(t))^2 + b(t)} = \frac{2}{\sqrt{b(t)}} \tan^{-1} \frac{\epsilon}{\sqrt{b}}. \tag{191}$$

Thus as $t \to t^*$ we get a segment of infinite length.

Our goal must then be an upper bound on dL/dt as given by (188). The problem is one of selecting an arbitrary curve C, and then deforming the and stretching the initial vorticity field such that the support volume is preserved, and arranging the vorticity to maximize the rate of lengthening of C. This problem is completely analogous to the variational problem for axisymmetric flow without swirl, but without nearly a much control over the geometry. We know only that every vortex line is to be an unknot.

The freedom inherent in 3D now allows not only streching of an unknot, but also twisting and folding. This allows a vortex loop to be mapoped into an equivalent volume such that the vorticity is doubled in intensity, see Figure 9. Clearly the main question arising now must be how to handle the fact that κ is no longer directly related to the total length of the unknot.

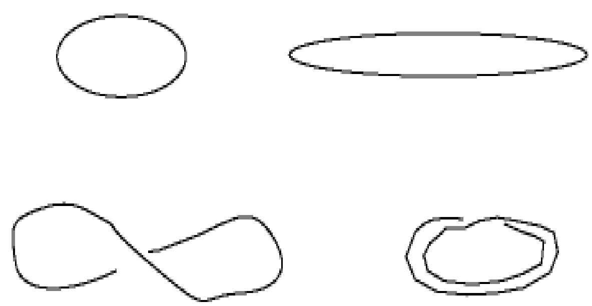

Figure 9. The stretch-twist-fold mechanism for amplification of vorticity (or magnetic field, i.e. of any frozen-in vector field). Note that the Figure omits reference to the deformation of the complement of support of the field.

It is worth mentioning a related question that arises in magnetostatics. Suppose a sphere is filled with an axisymmetric magnetic field consisting of unknots of field, i.e. all field lines are circles with a common axis. Does there exist a diffeomorphism which will reduce the total energy of the magnetic field to an arbitrarily small value? Since the analogous fluids question is to minimize the L^2 norm of the vorticity, the object here is maximal "anti=stretching" of vorticity. It turns out that such a diffeomorphism exists, see [1] p. 134. The trick is to stretch all but a small outer shell into a thin pencil shape, which is then wound up into the central space. The shell occupies the complement of this process, but this, along with the folding, can be estimated to contribute little energy, so the depletion comes mainly in forming the pencil shape.

9 Kinematics: the fast dynamo problem

We have seen that issues of global regularity might be attacked through an essentially kinematic argument, wherein vorticity is subject to certain constraints but is otherwise at our disposal to arrange to try to maximize vorticity growth. We conclude these notes with a discussion of a related problem in magnetohydrodynamics, namely the dynamo problem, and especially the kinematic dynamo problem.

Dynamo theory attempts to explain the origin of magnetic field in the cosmos, and the persistence of these fields in the presence of dissipation, through active dynamo excitation in a moving conducting fluid. For example, in the fluid core of the earth, convective motions of the electrically conducting (essentially iron) material is thought generate the magnetic field of the Earth through a dynamo process. The equation for the magnetic field in a homogeneous electrically conducting fluid takes the form

$$\mathbf{B}_t + \nabla \times (\mathbf{B} \times \mathbf{u}) - \epsilon \nabla^2 \mathbf{B} = \mathbf{b}_t + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} - \epsilon \nabla^2 \mathbf{B} = 0. \tag{192}$$

Here $\mathbf{B}$ is the solenoidal magnetic field, $\nabla \cdot \mathbf{B} = 0$. The parameter $\epsilon > 0$ is the inverse of a "magnetic Reynolds number",

$$\epsilon = (UL\mu\sigma)^{-1}, \tag{193}$$

where μ is the magnetic permeability, σ the magnetic conductivity, and $(\mu\sigma)^{-1} \equiv \eta$ is often referred to as the magnetic diffusivity. In most applications $\epsilon \ll 1$, and can be smaller than 10^{-10} in astrophysical applications.

The velocity field $\mathbf{u}$, which will be taken here to be divergence-free, is determined by dynamical considerations, involving the dynamical feedback of the magnetic field via the Lorenz "$\mathbf{J} \times \mathbf{B}$" force. This makes the full dynamo problem a branch of magnetohydrodynamics. In the spirit of the topological interests of these notes, we shall consider only the so-called kinematic dynamo problem, where the velocity field is regarded as a given motion. This may be a steady flow, or it may be periodic in time. We can think of the given $\mathbf{u}$ as the product of some dynamical system not involving a magnetic field. The kinematic dynamo problem is thus reasonable in the following sense: we can ask for the motions $\mathbf{u}$ which make a very small "seed" field grow indefinitely in time. So long as the field remains sufficiently small, the back-reaction of the field on the flow can be neglected. Another way to think of this in dynamics system terms is that we would like to find a $\mathbf{u}$ such that the zero $\mathbf{B}$ state of the system is unstable to small seed fields. If a system is known to have the property that all zero-field states are unstable in this way, such a system is a good candidate for acquiring a magnetic field.

Now (192) is formally equivalent to the viscous vorticity equation with ω replacing $\mathbf{B}$. The difference is of course that $\mathbf{B}$ is not the curl of $\mathbf{u}$. Nevertheless we can also envisage the frozen-in property of the vorticity and magnetic field as have e analogous kinematics, so the stretching, twisting, and folding of vortex tubes can be treated analogously as a kinematic dynamo problem at $\epsilon = 0$.

But we must make a careful distinction between $\epsilon \to 0$ and $\epsilon = 0$. If $\epsilon = 0$ and the field is frozen into the fluid, motions in 3D can produce small-scale, intense magnetic field which do not dissipate. However such fields can be rapidly dissipated for even a very small ϵ.

Since (192) is a linear equation for $\mathbf{B}$ given suitable boundary conditions we can imagine that the following eigenvalue problems could be posed for $\epsilon > 0$. In the case of steady $\mathbf{u}$, set $\mathbf{B}(\mathbf{x}, t) = e^{pt}\mathbf{b}(\mathbf{x})$. If there exist eigenvalues p such that the real part of p is positive, we say $\mathbf{u}$ is a kinematic dynamo. In the case of time-periodic $\mathbf{u}$, the existence of a Floquet exponent p with positive real part, so that the magnetic field acquires a factor e^{pT} over one time period T, implies dynamo action. Thus the search for kinematic dynamos involves looking at the dependence of the spectrum of a differential operator on the coefficients (given by $\mathbf{u}$) which define the operator.

Since our interest is in small ϵ it is useful to characterize kinematic dynamos by the behavior of the largest growth rate (the real part of p) for small ϵ Let this largest growth rate be $\sigma(\epsilon)$. them if $\sigma(\epsilon) \to 0$ as $\epsilon \to 0$ we say the dynamo is *slow*. If $\liminf \sigma(\epsilon) > 0, \epsilon \to 0$, we say that the dynamo is fast. A good reference to slow dynamo theory is [19]. The fast dynamo problem is treated discussed in [7], and also in chapter V of [1].

9.1 Beltrami fields as dynamos

We give an example now of a construction of a dynamo. We consider an infinite expanse of fluid and spatially periodic fields defined throughout the fluid. We will also consider fields on two distinct spatial scales, while solving (192) with ϵ taken to be of order unity. In the dimensional form of (192), $\mathbf{x}$ will denote the large scale variation. We then define
$yv = \mathbf{x}/\delta$, where $\delta \ll 1$ is a ratio of small to large length scales. We want to consider a steady, spatially-periodic velocity field on a $(2\pi)^3$ lattice in $\mathbf{y}$, of the form

$$\mathbf{u} = \delta^{-1/2}\mathbf{U}(\mathbf{y}). \tag{194}$$

The idea is to try to find an ordering so that $\mathbf{u}$ generates a magnetic field of the form

$$\mathbf{B} = \mathbf{B}_0(\mathbf{x}, t) + \delta^{1/2}\mathbf{B}_1(\mathbf{x}, \mathbf{y}, t) + O(\delta). \tag{195}$$

Thus we intend to show how small-scale velocity field can generate large scale magnetic fields.[4]

Because of the small-scale structure and the presumed form of the fields, the dominant terms in (192) give the following balance to leading order, (order $\delta^{-3/2}$):

$$\epsilon\nabla^2_{\mathbf{y}}\mathbf{B}_1 = -\mathbf{B}_0(\mathbf{x}, \tau) \cdot \nabla_{\mathbf{y}}\mathbf{U}. \tag{196}$$

We formally solve the last equation on the small scales:

$$\mathbf{B}_1 = \epsilon^{-1}{\nabla^2_{\mathbf{y}}}^{-1}\mathbf{B}_0(\mathbf{x}, \tau) \cdot \nabla_{\mathbf{y}}\mathbf{U}. \tag{197}$$

To obtain an equation for the large-scale magnetic field we average (192) over the small scales, to obtain

$$\frac{\partial \mathbf{B}_0}{\partial t} + \epsilon^{-1}\nabla_{\mathbf{x}} \times \langle \mathbf{B}_1 \times \mathbf{U} \rangle - \epsilon\nabla^2_{\mathbf{x}}\mathbf{B}_0 = 0. \tag{198}$$

[4] What has been called “asymptotology” is being used freely here. The discovery of the correct scaling for consistent asymptotic approximation can sometimes be quantified, but it remains largely an art. Fortunately it is an art that can be acquired, and the best way is through experience, i.e. trial and error.

Here $\langle \cdot \rangle$ indicates the averaging over $\mathbf{y}$.

Now (198) give us an expression which can be used to determine which spatially periodic. It turns out the spatially-periodic Beltrami field can be identified as especially effective dynamos. To illustrate this point consider the ABC flow with $A = B = C = 1$,

$$\mathbf{U} = (\sin y_2 + \cos y_3, \sin y_3 + \cos y_1, \sin y_1 + \cos y_2). \tag{199}$$

We see that

$$\langle \mathbf{B}_1 \times \mathbf{U} \rangle = -\mathbf{B}_0. \tag{200}$$

Problem 9 *Verify* (200).

Thus we obtain a mean magnetic field equation,

$$\frac{\partial \mathbf{B}_0}{\partial t} - \epsilon^{-1} \nabla_{\mathbf{x}} \times \mathbf{B}_0 - \epsilon \nabla_{\mathbf{x}}^2 \mathbf{B}_0 = 0. \tag{201}$$

Setting $\mathbf{B}_0 = e^{i\mathbf{k}\cdot\mathbf{x}+pt}\Gamma$, where Γ is a constant vector, we see that

$$p\Gamma - i\epsilon^{-1}\mathbf{k} \times \Gamma + \epsilon k^2 \Gamma = 0. \tag{202}$$

Taking $\mathbf{k}\times$ the last equation, and solving the system of two equations, we obtain the "dispersion equation"

$$(p + \epsilon k^2)^2 = \epsilon^{-2} k^2. \tag{203}$$

We see that positive p exist when $\epsilon^{-1} k > \epsilon k^2$.

We point out that on the scale spatial scale of the velocity field the appropriate magnetic Reynolds number is $U\delta^{-1/2}\delta L/\eta = \delta^{1/2}\epsilon^{-1} \ll 1$. Thus the velocity eddies are very diffusive, and this is built into the asymptotic "smoothing" method we have used. While we have established that this Beltrami field is a dynamo in the space of periodic fields, we can say nothing at this point about the fast dynamo property. In fact the 111 Beltrami flow (199) is believed to be a fast dynamo, but the $\epsilon \to 0$ must be studied numerically. It is not accessible by asymptotic smoothing methods.

9.2 Dynamo action by a cellular flow for small ϵ

To illustrate analysis of dynamo action by flows with simple topology, in the limit of large R, we discuss no a simple cellular flow having the velocity field

$$\mathbf{u} = (\mathbf{u}_H, \psi) = (\psi_y, -\psi_x, \psi), \ \psi = \sin x \sin y. \tag{204}$$

Notice that $\nabla \times \mathbf{u} = \mathbf{u}$, we are dealing with a Beltrami field. In fact this flow is equivalent to "two-thirds" of the 111 ABC flow. This can be shown by dropping the terms in z and performing a rotation, translation, and scale change of the coordinates. The magnetic field associated with such a cellular flow may be taken to have the form

$$\mathbf{B}(x, y, z, t) = e^{pt+ikz}(\mathbf{b}_H, B(x, y)). \tag{205}$$

Note that the z-variation has been separated out since $\mathbf{u}$ is independent of z. The idea now is to insert this expression for $\mathbf{B}$ as well as (204) into (192) and then study the eigenvalue problem as $\epsilon \to 0$.

As a preliminary to this we consider the simple steady problem which results when $p = k = 0$ in (205). It is not difficult to show that in this case the x, y-magnetic field components $\mathbf{b}_H$ are given by $A_y, -A_x)$, where A is determined by the following advection-diffusion equation:

$$\mathbf{u}_H \cdot \nabla A - \epsilon \nabla^2 A = 0. \tag{206}$$

We want to solve this equation in the square bounded by $(0,0), (0,\pi), (\pi,\pi), (0,\pi)nd.$ Consider the boundary conditions corresponding to a mean magnetic field (1,0). In that case we may assign the values $A = -\pi/2$ and $A = \pi/2$ to the lines $y = 0$ and $y = \pi$ respectively. Along the two vertical boundaries we have $\partial A/\partial x = 0$ since A must be an even function with respect to these lines.

We consider this boundary problem now for small ϵ. Something analogous to the constant vorticity property of closed contour steady Euler flows in Prandtl-Batchelor theory happens here. The function A tends to constant over the square varies only in boundary layers near the walls. By symmetry this constant must be zero.

The boundary-layer equations are similar on each segment of the boundary. On the segment $y = 0, 0 \le x \le \pi$ for example, we have

$$\frac{\partial}{\partial y}\Big|_x = u\frac{\partial}{\partial \psi}, \frac{\partial}{\partial x}\Big|_y = \frac{v\partial}{\partial x}\Big|_\psi - v\frac{\partial}{\partial_\psi}\Big|_x. \tag{207}$$

In the boundary layer the fluid speed $q \sim u$ since v is small, and here on the segment in question we see that q is a function of x alone. Thus the boundary-layer equation on this segment, and indeed on all segments, takes the form

$$q^{-1}\frac{\partial A}{\partial s} - \frac{\partial^2 A}{\partial \xi^2} = 0, \xi = \psi/\sqrt{\epsilon}, \tag{208}$$

where s is distance along the wall. Note that the boundary layer thickness is of order $\epsilon^{1/2}$. Setting $qds = d\sigma$ yields the heat equation:

$$\frac{\partial A}{\partial \sigma} - \frac{\partial^2 A}{\partial \xi^2} = 0. \tag{209}$$

At the values of σ corresponding to the corners, we require that A be continuous when $\xi > 0$. Thus we get a circulating boundary layer satisfying continuity at these values of σ and boundary conditions of either $A = \pm\pi/2$ or $\frac{\partial A}{\partial \xi} = 0$ at $\xi = 0$ depending upon the segment.

A closed-form solution of this problem is possible, see [7], sec. 5.5.1, but we omit details, out purpose here being simply to indicate the nature of the boundary-layer structure in the problem in the steady case.

Note the physical significance of this structure: the magnetic field is excluded from the bulk of the domain and confined to the perimeter. This phenomenon is called *flux expulsion*.

In the general case we write $\mathbf{b}_H$ satisfies

$$(P + \mathbf{u}_H \cdot \nabla)\mathbf{b}_H = \mathbf{b}_H \cdot \nabla \mathbf{u}_h + \epsilon \nabla^2 \mathbf{b}_H, \tag{210}$$

where $P = p + ik\psi$ and ∇^2 here is in x, y. The component B now follows from the solenoidal condition, $\nabla_H \cdot \mathbf{b}_H = -ikB$. The magnetic structure is still confined to boundary layers, so dynamo action happens there as well. An essential difference between the steady and unsteady problems occurs in the corner regions, and the difference underscores the importance of hyperbolic stagnation points on flow kinematics. The term P in (210) modifies the continuity conditions at the corners. One sees that an integral of the form $P \int q^{-1} ds$ will intervene, which gives the time of passage of a fluid particle through the hyperbolic flow at the stagnation point. For small ψ, the time of passage goes as $-\log \psi$ and this affects the calculation of dynamo action. It was shown by Soward that when one looks at the maximal growth rate, maximized over k, and small ϵ, this rate goes as $\log \log \epsilon^{-1} / \log \epsilon^{-1}$. Since this last quantity goes to zero as $\epsilon \to 0$, we see that the cell is not a fast dynamo, but the rate of decrease is so slow that there is little practical difference.

The mechanism of dynamo action is interesting in this example, for it occurs entirely in boundary layers. Suppose that the spatial mean of the magnetic field is momentarily concentrated in the x-directed flux. Then the x, y velocity components in the vertical layers tend to pull out "tongues" of flux as loops extending up into the vertical layers. These structures do not change the mean field because the flux cancels out when integrated across the layer. However the z- component of velocity also has its effect, and this is a shear which, because of the z-periodicity of $\mathbf{B}$, causes mean magnetic flux in the y-direction to be created. (see the figure on page 140 of [7]. This y-flux is now acted upon by the velocity field to pull out tongues in the x-direction, which with shear creates mean flux in the x-direction, and the cycle is complete. For sufficiently small ϵ the mean flux is increased over a cycle, and dynamo action occurs.

9.3 The SFS fast dynamo

We now describe a map of the cylinder $[0,1]^2 \times (-\infty, +\infty)$ in R^3 which is meant to model the mechanism of dynamo action obtained above within the boundary layers of the periodic cell. Iterations of the map will produce Lagrangian chaos and lead to fast dynamo action. The map is defined by

$$(x, y, z) \to \begin{cases} (2x, y/2, z + \alpha f(y/2)), & \text{if } 0 \le x \le 1/2, \\ (2 - 2x, 1 - y/2, z + \alpha f(1 - y/2)), & \text{if } 1/2 \le x \le 1. \end{cases} \tag{211}$$

The action of this map for $\alpha = 1$ is shown on page 68 of [7]. It consist of a "stretch-fold" or baker's map, and a shear. We set

$$\mathbf{B} = e^{2\pi ikz}(b(y), 0, 0), \tag{212}$$

We can then represent the action of the map on the magnetic field, in the absence of diffusion, by the map T acting on $b(y)$:

$$Tb = 2\text{sign}(1/2 - y)e{-}2\pi ik\alpha(y - 1/2)b(\tau(y)), \tag{213}$$

where $\tau(y)$ the "tent map", $\tau(y) = \min(2y, 2 - 2y)$. We set

$$b(y) = \sum_{n=-\infty}^{+\infty} b_n e^{2\pi iny}. \tag{214}$$

and represent the effect of diffusion by the map on Fourier coefficients

$$H_\epsilon : \quad b_n \to b_N e^{-4\pi^2\epsilon(n^2+k^2)}. \tag{215}$$

We call this map the SFS map, for stretch-fold-shear. It is easily simulated and one finds for small ϵ that large, rapidly varying $b(y)$ is created after a few iterations. To measure the overall effect of the map in the average field, we track the mean flux

$$\Phi = \int_0^1 b(y)dy. \tag{216}$$

It is found that positive growth rate of Φ occurs for α sufficiently large. We remark that in the case $\epsilon = 0$ it is difficult to obtain numerical accuracy for the direct calculation of ϕ because of the small-scale chaotic structure of the field. However it can be shown that the map which is the formal adjoint to T in the relevant L^2 norm has smooth eigenfunctions whose associated eigenvalues determine growth rates. So the adjoint map is therefore used to compute overall growth rate when $\epsilon = 0$.

9.4 General properties of fast dynamos

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A necessary ingredient of fast dynamo action is a chaotic Lagrangian structure for fluid orbits. This is achievable in steady flows in three space dimensions, or in unsteady flows in 2D. The latter are most easily studied numerically. An example is the flow

$$\mathbf{u}(x, y, t) = 2\cos^2(t)(0, \sin x, \cos x) + 2\sin^2(t)(\sin y, 0, -\cos y). \tag{217}$$

Note that this flow superimposes two one-dimensional Beltrami flows modulated in time.

All fast dynamos produce large fluctuating fields for small ϵ. In fact most of the magnetic energy is in these fluctuations. For any positive ϵ, the field of maximal growth rate is obtained as an eigenfunction (of a Floquet problem in the case of (217)). As $\epsilon \to 0$, these become acquire indefinitely irregular structure, and are called *strange eigenfunctions*. It was possible to circumvent this difficulty with the SFS model because of the existence of the well-behaved adjoint, but for (217) the adjoint is not any simpler in its effect.

It is natural to speculate that calculations at zero ϵ could be used to extract the growth rate which is obtained by extrapolating calculations for $\epsilon > 0$ to $\epsilon = 0$. As the use of the flux function for the SFS map suggests, some sort of projection of magnetic structure onto a mean field is needed. The equality of the growth of a suitable projected field, to the extrapolated growth rate of the diffusive eigenfunction, can be demonstrated numerically in many cases, but in only a very few can the relation be proved to occur.

Fast dynamo theory is thus a kind of kinematic realization of the kind of structure one imagines the vorticity field to have in fully developed turbulence at large Reynolds numbers, as well as in the 3D IVP for Euler flows.

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