Notes on traffic flow

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1 The modeling of automobile traffic

When one thinks of modeling automobile traffic, it is natural to reason from personal experience and to visualize the car and driver as a coupled system, the driver responding to the surrounding vehicles and operating the car to make it become a part of the flow of freeway and city traffic. Thus the traffic is not just a mechanical process but one in which human decisions are involved, decisions which we have all experienced and can understand.

In our analysis of traffic we shall however step back from this personal view to take a broader perspective. Think of a traffic helicopter pilot looking down on the NYC highway grid. Looking at four miles of highway, the pilot will see a line of cars moving with various speeds. On some stretches, the traffic may be light and fast, on other stretches heavy and slow. To this observer the individual vehicles are not as important as the sense of overall *flow* of the cars. The reason why the cars in the lighter traffic move faster is clear to any driver, but to the observer in the helicopter it seems to be a property of the spacing of the cars. The closer they are together, the slower they move. Models of traffic flow try to exploit these observations and use them to formulate a set of assumptions which produce models which can be used to try to understand the peculiar and often frustrating occurences of daily driving. Have you ever experienced an hour of creeping traffic on a freeway, only to realize upon passing a patch of water on the road that this puddle is the cause of the problem, with every driver slowing down as they pass it? What is the effect of the closing of one lane one a four-lane throughway going to do to the traffic at rush hour? (We all know too well.) One ought to be able to understand some of the large effects of seemingly small causes.

In the picture just suggested, the cars are viewed in the large, almost as a moving gas or liquid. This kind of picture we will call a *continuum model of traffic flow*. We shall spend much of our time working from such a point of view. There is however another body of traffic theory based upon the point of view of the individual driver responding to surrounding traffic- just the way we would naturally want to think about driving. This kind of study is called *car-following theory*. We shall also look at some examples of this approach, which can be thought of as analogous to studying a gas by analyzing the motion of the individual molecules.

2 Formulation

Ultimately the traffic engineer is interested in how fast cars move through the traffic grid. Every car has a speedometer, and we all want to know how long it will take to go from A to B. Certainly one of the main quantitative measures of traffic is the speed of the cars in the traffic. Consider, for the sake of argument, a one-lane highway with cars in a line moving in the same direction. Since there is no passing, and cars cannot move through each other, the order of the cars is preserved, although they can move at slightly different speeds. Let the velocity of car "i'' be u_i. If the x-axis coincides with the road and the position of this car is $x_i(t)$ at time t, then the calculus tells us how u_i and x_i are related: u_i is the derivative with respect to time of x_i. Thus

$$u_i = \frac{dx_i}{dt} \tag{1}$$

Any discussion of traffic on our single-lane road must deal with a collection of vehicles, with positions $x_i(t), i = 1, 2, \ldots, N$ and velocities $u_i = \frac{dx_i}{dt}, i = 1, 2, \ldots, N$. The continuum approach to traffic takes the view that this collection of discrete objects should be replaced by a "moving continuum", a kind of fluid of vehicles. Such a fluid has a velocity at every value of x and at every time, and so we may define a *velocity field* by a function $u(x, t)$. The idea is that the variation of $u(x, t)$ with x should be on a scale of length (say a hundred yards) which is large compared to the size of a typical vehicle. Thus the value of $u(x, t)$ at a certain time t^* and a certain position x^* on the road should be the velocity of cars on that particular part of the road at that time.

If we know the velocity field for our road, how do we find the movement of an individual car? First we must specify the car. One way to do that is to choose a particular time, say $t = t_0$, and a particular position on the road, say $x = x_0$, and identify a car as being at that spot at that time. If we then want to know where this car is located at times $t > t_0$, we must use our knowledge of the velocity field, which tells us how fast any car is going when at position x and time t. Thus if $x(t)$ is the position of our car, we know that $x(t_0) = x_0$ but also that

$$\frac{dx}{dt} = u(x(t), t). \tag{2}$$

This last equation is the crucial one, since it relates the overall velocity field to the function $x(t)$ for the particular car which was located at x_0 at time t_0. We sometime will call $x(t)$ the *Lagrangian coordinate* of the car. This name pays respect to the French mathematician Lagrange (1736-1813), who introduced the description of a fluid by tracking the motion of individuals particles of the fluid. Also, $u(x, t)$ is sometimes called the *Eulerian* velocity field, after the Swiss mathematician Euler (1707-1783), who introduced the description of fluid motion using a velocity field.

Note that the problem of locating the position of our car, summarized as

$$\frac{dx}{dt} = u(x(t), t), \quad x(t_0) = x_0, \tag{3}$$

where $u(x,t)$ is a given function, amounts to solving an ordinary differential equation of first order with an initial condition at the time t_0.

Problem 1: Consider the units of x to be in miles. On the stretch of road $0 < x < 4$ cars are accelerating from a red light, and the velocity field is found to be $u(x,t) = 10x + 30t$ miles per hour where $t > 0$ is measured in hours. What is the Lagrangian coordinate of the car which was located at $x = 1.5$ at time $t = 0$? To answer this we must solve

$$\frac{dx}{dt} = 10x + 30t, \quad x(0) = 1.5. \tag{4}$$

Using the integrating factor e^{-10t}, we have $\frac{d}{dt}(e^{-10t}x) = 30te^{-10t}$. Integrating by parts, we get $e^{-10t}x = -(.3 + 3t)e^{-10t} + C$ or $x = -(.3 + 3t) + Ce^{10t}$. From our initial condition $x(0) = 1.5 = -.3 + C$, or $C = 1.8$. Thus $x(t) = -(.3 + 3t) + 1.8e^{10t}$.

We shall make frequent use of the $x - t$ plane in describing traffic flow. In figure 1 we show the velocity field just discussed, represented by the family of vehicle trajectories, each curve being the path of a car. (Note: the text sometimes uses the very nonstandard $t - x$ diagram instead of the standard $x - t$ diagram we use throughout here. Watch for this in your redading.)

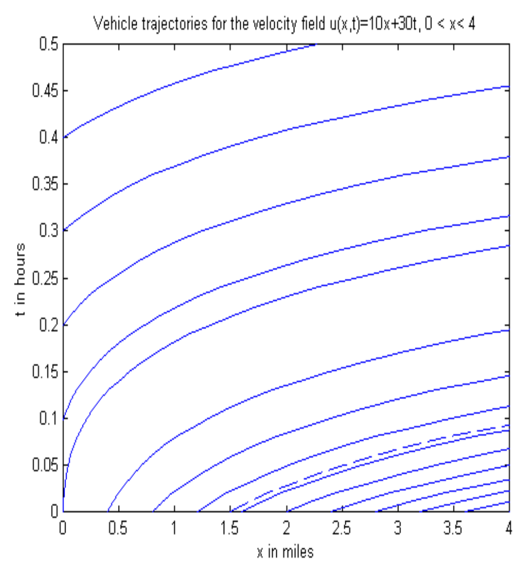

Figure 1. The vehicle trajectories for the velocity field $u(x,t) = 10x + 30t$, $0 < x < 4$. The dashed line is the path of the vehicle initially at $x = 1.5$ miles.

If a vehicle is traveling at constant velocity u, then in the $x - t$ plane its path is the straight line $x = u(t - t_0) + x_0$.

If we know the trajectories of all of the vehicles in a traffic flow, then we ought to be able to determine the velocity field of the flow. This is the problem inverse to the one we just solved, which was to find the car trajectory given the velocity field.

Problem 2 Let the cars' trajectories be given by $x = t^2 + 2tx_0 + x_0$. Note that $x(0) = x_0$, identifying the parameter x_0 as the initial position. Find the velocity field for this flow. To do this first compute the velocity, then use the two equations to eliminate x_0. Thus we have $\frac{dx}{dt} = u = 2t + 2x_0$, where the first equation tells us that $x_0 = \frac{x-t^2}{1+2t}$. Therefore $u(x,t) = 2t + \frac{2x-2t^2}{1+2t} = \frac{2t+2t^2+2x}{1+2t}$.

We point out that the parameter which determines which car we pick need not be the initial position.

Problem 3 let a family of vehicle trajectories be given by $x = ae^t + a^2, a > 0$. Find the corresponding velocity field $u(x,t)$. We have $u = ae^t$ and $a = (\sqrt{e^{2t} + 4x} - e^t)/2$. Thus $u(x,t) = e^2(\sqrt{e^{2t} + 4x} - e^t)/2$.

Problem 4 (See text, problem 57.6, page 264). Since every car has constant acceleration β, and starts (at $t = 0$ with zero velocity, we have $u = \beta t = \frac{dx}{dt}$. Since $x(0) = \beta$, we obtain $x = \beta t^2/2 + \beta$. To get the velocity field we elminate β between $u = \beta t$ and $x = \beta t^2/2 + \beta$. This gives $u(x,t) = \frac{xt}{1+t^2/2}$.

3 Traffic density

The second basic measure of traffic in a continuum model, in addition to the velocity field, is the *traffic density.* We again imagine a one-lane road with cars spread along it. The traffic density on this road associated with a given position x and time t, is the average number of vehicles per unit legnth of road at the position and time specified. Clearly to measure a density we need a stretch of road with enough cars on it to allow a reasonable statistical average. At the same time, we want to talk about the spatial variation of traffic density along the road, so the length over which we average should not be too long either, or else we will be getting to the natural scale of variation of the density. (You should read section 58 of the text for further discussion of this issue.)

We will use the traditional symbol for fluid density, namely ρ, for the traffic density. Thus $\rho(x,t)$ is the average number of cars per unit length at the position x and time t.

If all vehicles have length L (or else L is a good average length, and the spacing (or average spacing) between the cars is d, then each vehicle takes up $L + d$ units of road, so that approximates $\frac{1}{L+d}$ vehicles will be present per unit length of road. Thus the constant density of the traffic in this case is $\rho = \frac{1}{L+d}$.

4 Traffic flux

Again we think of our one-lane road, now having traffic with a certain density and velocity field. Another thing we need to think about is the common usage of the term *traffic flow.*We mean by this the rate at which What this is refering to is the rate at which cars an observer on the edge of the road, i.e. the *number of cars per unit time which cross a given point on the road*. We have seen the line accross a road that counts passing vehicles. This is being used to determine the traffic flow. Actually we shall prefer to use another term: *traffic flux*. The flux of cars is the wame as the flow (or better, the flow rate) of cars–i.e. the number of vehicles going by per unit time.

A key equation: Flux equals velocity times density If there are 100 cars per mile on a road, and each car is going 60 mph, then in one hour 60 miles worth of cars will pass an observer at the side of the foad, or $60 \times 100 = 6000$ cars per hour. This is the flux in this example., with $u = 60$ miles per hour and $\rho = 100$ vehicles per mile. The flux is $\rho u = 60 \times 100$ vehicles per hours (in the units, the "miles" cancel out).

We shall use the symbol q for flux. The flux is another key function which in general will depend, like u and ρ, upon x and t. thus

$$q(x,t) = \rho(x,t)u(x,t), \tag{5}$$

Problem 5 Assume that cars cannot leave or enter a one-way, two lane thoroughfare. At some point one lane is closed. After some confusion, the situation settles down to a steady state. The conditions well before the lane closing are seen to be constant and uniform; both lanes have the same density ρ_1 and the same velocity u_1. Note that ρ_1 is the density in each lane. Well after the closing of one lane, conditions are again uniform on the single lane with $u = u_2$ and $\rho = \rho_2$. What relation must exist between ρ_1, ρ_2, u_1, u_2? Since there are two lanes before the closing, the flux of cars on the throughfare there is $q_1 = 2\rho_1 u_1$. Well downstream of the closing the flux is $q_2 = \rho_2 u_2$. Since everthing is steady, and cars cannot leave or enter, the flux into the closing region must equal to that out of the region. Thus we have $2\rho_1 u_1 = \rho_2 u_2$.

Thus if the speed stays the same ($u_2 = u_1$), the density must double. In fact, higher density generally forces drivers to go slower, which drives the density up even further in order to preserve flux, leading to extreme slowdowns. To really exmine this problem, however, we need to study how flux and density are related through "conservation of vehicles" in roads without entrance or exit.

5 Conservation of the number of vehicles

Again we consider our 'bare" one-lane road (no entrances or exits). If we select some stretch of the road, between points $x = A$ and $x = B > A$ say, we know that the number of cars found to lie between A and B at some time t will in general depend upon the time t. If more cars flow into the segment AB than flow out of it, the number of vehicles within the segment will increase, and similar if more flow out than in, it will decrease. We can express this mathematically in terms of the *flux* at A and B. Namely, the rate of change of the number of vehicles in the segment, with respect to time, should equal the difference in flow rate or flux. If $N_{AB}(t)$ is this number of vehicles, then

$$\frac{dN_{AB}}{dt} = -q(B,t) + q(A,t). \tag{6}$$

On the other hand we know that N_{AB} can be comuted from the density by integration:

$$N_{AB}(t) = \int_A^B \rho(x,t)dx, \tag{7}$$

Thus we can rewrite our relation as

$$\frac{d}{dt}\int_A^B \rho(x,t)dx = -q(B,t) + q(A,t). \tag{8}$$

We say that this last result is a *global conservation law* for the vehicles on the road. Note that the signs on the right are consistent. If $q(B,t) > q(A,t)$ the more calls flow out than in, so N_{AB} will decrease in time.

For example, in problem 5 we considered the effect of a lane closing, and assume that the road had "settled into a steady state"; we then deduced that the fluxes were the same. We can see from our conservation law (8) that $q_(B,t)-q(A,t)$ whenever ρ becomes independent of time. This is because

$$\frac{d}{dt}\int_A^B \rho(x,t)dx = \int_A^B \frac{\partial\rho}{\partial t}dx, \tag{9}$$

since A, B are constants. Note the use of the partials on the right, since ρ also depends upon x.

The global conservation law implies a *local* conservation law, expressing the conservation of vehicle number on any stretch of road sufficiently long to allow us to assign a meaningful velocity and density function. To get the local relation we use the fundamental theorem of the calculus:

$$\int_A^B \frac{\partial q(x,t)}{\partial x}dx = q(B,t) - q(A,t). \tag{10}$$

Note that we have used a partial derivative with respect to x here, since q depends on both x and t. Otherwise this is the calculus I theorem.

Using (8),(9), and (10), we have

$$\int_A^B [\frac{\partial\rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x}]dx = 0. \tag{11}$$

Now this relation holds over an interval AB. We now use the following result (which we do not prove here): Let $f(x)$ be continuous on some closed interval $[\alpha, \beta]$ and assume that

$$\int_A^B f dx = 0 \tag{12}$$

for any interval $AB \in (\alpha, \beta)$. Then $f = 0$ for $\alpha < x < \beta$.

Using this result, we see that (11) implies that

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0. \tag{13}$$

Using (5) we can write (13) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0. \tag{14}$$

Problem 6 let $x_A(t)$ be the position of one car on our one-lane road, and $x_B(t)$ be the position of another car many miles ahead. Then the number of cars between cars A and B is a constant, independent of time.

Of course this is obvious physically, but we can show it using calculus:

$$\frac{d}{dt}\int_{x_A(t)}^{x_B(t)} \rho(x,t)dx = \int_{x_A(t)}^{x_B(t)} \frac{d\rho(x,t)}{dt}dx + \rho(x_B(t),t)\frac{dx_B(t)}{dt} - \rho(x_A(t),t)\frac{dx_A(t)}{dt}. \tag{15}$$

But the right-hand-side of the last equation is just

$$\int_{x_A(t)}^{x_B(t)} \frac{d\rho(x,t)}{dt}dx + q(x_B(t),t) - q(x_A(t),t) = 0 \tag{16}$$

by (8).

6 Velocity as a function of density

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The equation we now have for vehicle conservation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \tag{17}$$

is one relation involving two unknowns. Conventionally, we would need another relation to close the system in two unknowns. A major assumption that is often made by traffic modelers is that *velocity may be reasonably assumed to be a function of the density alone.* That is, we can assume $u = u(\rho)$ and our equation becomes a relation in ρ and its derivatives:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u(\rho))}{\partial x} = 0, pde) \tag{18}$$

Such an equation is called a *partial differential equation (PDE) of first order.* It is a PDE because of the two variables involved, x, t, and the partial differentiations with respect to these variables. It is a first-order equation because only first partials are involved. To see this set $F(\rho) = \rho u(\rho)$. Then the equation (**??**) can be written

$$\frac{\partial \rho}{\partial t} + F'(\rho)\frac{\partial \rho}{\partial x} = 0. \tag{19}$$

We will focus later on how to solve this equation once $u(\rho)$ is given. For the moment our concern is whether or not this assumption is justified, and then what the function $u(\rho)$ should be.

On a single-lane open road this assumption seems to be fairly reasonable. An isolated car tends to have a maximum velocity of travel, either the result of speed limits or road conditions or driver caution, call it u_{max} Then for our function $u(\rho)$ we should take $u(0) = u_{max}$. We know that traffic speeds tend to go down with increasing traffic density, so we should assume that $du/d\rho < 0, \rho > 0$. Also there is surely a density, bumper to bumper traffic say, where the speed is essentially zero. Call this density ρ_{max}. If L is average car length, we could take $\rho_{max} = 1/L$. One widely used relation is

$$u(\rho) = u_{max}(1 - \rho/\rho_{max}). \tag{20}$$

Most of our discussion will concern the model of traffic flow which results from using (20).

Another argument might go as follows: assume that a deceleration rate A is tolerable to a driver, and therefore the driver will tend to stay a distance d behind a car so that if the car suddenly slowed to half (say) of its speed, the driver would be able to avoid a collision. The time it takes to decelerate from speed u to speed $u/2$ is $T = \frac{u}{2A}$, and the shortening of the distance between the cars to zero in this time implies that

$$\int_0^T (u - At)dt = d + uT/2, \tag{21}$$

giving $d = u^2/(8A)$. If the average car length is L, then $1/(d + L)$ should be the density when moving at speed u. Thus

$$\rho = \frac{1}{L + u^2/(8A)}, \tag{22}$$

, or

$$u = K\sqrt{\frac{1}{L\rho} - 1}, K = \sqrt{8AL}. \tag{23}$$

Since this relation goes to $+\infty$ as $\rho \to 0$, we must cut it off when $u = u_{max}$. Thus

$$u = \begin{cases} u_{max}, & \text{if } 0 < \rho < \rho_{min}, \\ \sqrt{8AL}\sqrt{\frac{1}{L\rho} - 1}, & \text{if } \rho > \rho_{min}. \end{cases} \tag{24}$$

Here

$$\rho_{min} = L^{-1}[1 + \frac{u_{max}^2}{8AL} + 1]^{-1}. \tag{25}$$

In figure 2 we compare relations (20) amd (24) for some typical numbers.

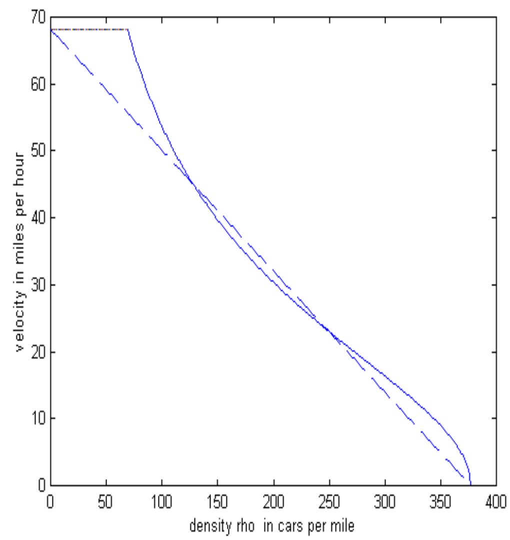

Figure 2. Velocity as a function of density using (20)(dotted line) and (24) (solid line). We take $L = 14 feet$, $A = 20\ ft/sec^2$, $u_{max} = 100\ ft/sec$.