Applied Math II Spring 2003

The Riemann function for the wave equation in two space dimensions

We have established that the Riemann function in three dimensions is given by

$$R^{(3)}(\mathbf{x},t) = \frac{\delta^{(1)}(t - \frac{\sqrt{x^2+y^2+z^2}}{c})}{4\pi c^2\sqrt{x^2+y^2+z^2}}$$

We use the "3" superscript to indicate three dimensions. The 3D delta function is a product of three 1D delta functions:

$$\delta^{(3)}(\mathbf{x}) = \delta^{(1)}(x)\delta^{(1)}(y)\delta^{(1)}(z).$$

The $\delta^{(1)}$ in the above Riemann function is a 1D delta function indicating the function obtained as one crosses the surface $S:\ R^2 \equiv x^2+y^2+z^2 = c^2t^2$ radially. When this delta function appears in an integral times a test function, it will pick out the integral of the test function over S, but there are some factors which appear which come from properties of $\delta^{(1)}$

We first consider these properties.

Property 1: If $\phi(x)$ is a test function and c is a constant (and dropping the superscript "1" for the moment), then

$$\delta(cx) = \frac{1}{|c|}\delta(x).$$

To prove this, let ϕ be any test function and $a > 0$. Then with $u = cx$

$$\int_{-a}^{+a} \phi\delta(cx)dx = \frac{1}{c}\int_{-ac}^{+ac} \phi(u/c)\delta(u)du = \frac{1}{|c|}\phi(0).$$

A special case of this is $\delta(-x) = \delta(x)$.

Property 2: Let $f(x)$ be a twice-differentiable function with at most a finite number of zeros, $x_1, \ldots x_N$, all of which are simple. Then

$$\delta(f(x)) = \sum_1^N \frac{1}{|f'(x_i)|}\delta(x - x_i).$$

To prove this consider

$$S \equiv \int_{-\infty}^{+\infty} \phi(x)\delta(f(x))dx.$$

Now contributions to S come only within small intervals around each point where $f(x)$ vanishes, i.e. the N simple zeros. Set $df/dx = F(x)$. Then we have formally the change of variable

$$S = \int \frac{1}{F(x)}\phi(x)\phi(f)df.$$

If f is increasing with x at a zero, the contribution at that zero will be $\frac{1}{F(x_i)}\phi(x_i)$ If f is decreasing at the zero then we will get The negative of the last expression. Since $F(x_i) > 0$ when increasing and < 0 when decreasing, we get $\frac{1}{|F(x_i)|}phi(x_i)$ at every zero. This establishes that we get the summation as claimed.

We now obtain the Riemann function $R^{(2)}$ for two space dimension, from $R^{(3)}$, by a technique known as the *method of descent*. Let $L^{(3)}u = \partial_t^2 u - c^2\nabla^2 u$ denote the wave equation on three space dimensions. We know that

$$L^{(3)}R^{(3)} = \delta^{(3)}(\mathbf{x}) = \delta^{(1)}(x)\delta^{(1)}(y)\delta^{(1)}(z).$$

We now claim that $R^{(2)}$, the Riemann function in 2D, can be obtained as

$$R^{(2)}(x,y,t) = \int_{-\infty}^{+\infty} R^{(3)}(x,y,z,t)dz.$$

Indeed we note that

$$\int_{-\infty}^{+\infty} L^{(3)}R^{(3)}(x,y,z,t)dz = \int_{-\infty}^{+\infty} L^{(2)}R^{(3)}(x,y,z,t)dz$$

$$= L^{(2)}R^{(2)} = \int_{-\infty}^{+\infty} \delta^{(1)}(x)\delta^{(1)}(y)\delta^{(1)}(z)dz = \delta^{(1)}(x)\delta^{(1)}(y),$$

as required.

To do the integral, we hold t and $x^2 + y^2 = r^2$ fixed and consider the integral in the form

$$\int_{-\infty}^{+\infty} \frac{\delta^{(1)}(f(z))}{4\pi c^2 g(z)} dz,$$

where $f(z) = t - \sqrt{r^2 + z^2}/c$ and $g(z) = \sqrt{r^2 + z^2}$. There are zeros of $f(z)$ only if $r^2 < c^2t^2$, in which case there are exactly two simple zeros, $z = \pm\sqrt{c^2t^2 - r^2}$.

Wec leave it as an exercise to show that the above properties of the one-dimensional delta function enaable this integral to be evaluated, with the result that

$$R^{(2)}(x,y,t) = \frac{1}{2\pi c} \frac{H(ct - r)}{\sqrt{c^2t^2 - r^2}}.$$

Here H is the Heaviside function $H(x) = 0, x \leq 0, H = 1, x > 0$.

This expression for $R^{(2)}$ shows that, unlike the 3D Riemann function, the signal is not confined to $r = ct$. In fact on that curve the signal is infinite as in 3D, but there is a non-zero but decaying signal for $ct > r$. At any given time therefore, a given point (x, y) can be affected by data within the entire disc $R < ct$. Thus the domain of dependence is not just a boundary $r = ct$, as in the 3D case, but fills the domain. Thus Huygens' principle fails to obtain in two dimenions. In fact this is the case for all even dimensions ≥ 2. Huygens' principle can be shown to hold in all odd dimensions ≥ 3.