Intro. to Math. Modeling   PROBLEM SET 4   Due February 14, 2005

1. A population is divided into three age groups: 0-14 years, 15-39 years, 40 or older, as described in class. In that model, we take $b_1 = b_3 = 0, b_2 = .4, d_1 = .03, d_2 = .1, d_3 = .3$. Show that the growth matrix A is

$$A = \begin{pmatrix} .9053 & .4 & 0 \\ .0647 & .8640 & 0 \\ 0 & .0360 & .7 \end{pmatrix}.$$

Computation shows that (to four decimals)

$$A^{50} = \begin{pmatrix} 5.5549 & 12.1537 & 0 \\ 1.9648 & 4.2990 & 0 \\ 0.2040 & 0.4462 & 0.0000 \end{pmatrix}, A^{100} = \begin{pmatrix} 54.7367 & 119.7607 & 0 \\ 19.3613 & 42.3614 & 0.0000 \\ 2.0097 & 4.3971 & 0.0000 \end{pmatrix}.$$

(i) From these results, we can estimate an age distribution vector of the form $\begin{pmatrix} a \\ 1 \\ b \end{pmatrix}$. What are a, b? (If you want, take the starting population as $\begin{pmatrix} 1000 \\ 1000 \\ 1000 \end{pmatrix}$.

(ii) Calculate the growth rate r, where the population grows by the factor $1 + r$ per year. (recall $A \cdot \begin{pmatrix} a \\ 1 \\ b \end{pmatrix} \approx (1 + r) \begin{pmatrix} a \\ 1 \\ b \end{pmatrix}$).

2. From the population data handed out in class, auswer the following:

(a) The birth and death rates for the year 2000 were 14.4 and 8.5 respectively per 1000 population. Justify this claim. Estimate the average growth rate of the population from 1950 to 200, i.e. find r wihere $(1 + r)^{50}$ is the factor by which the total population increased. Comment on difference betwee r and $b - d$.

(b)Estimate the birth rates $b_m, m = 3, 4, 5$ and death rates $d_m, m = 6, 7, 8$, for the eight age groupings shown in the year 2000, as follows. Compute the number of births in the age group by multiplying the total births by the fraction of the population estimated from the projected age distribution. Thus for example $b_3 = \frac{(.457)(4{,}058{,}814)}{(.143)(282{,}421.906)} = .0459$. Similarly determine the death rates. Assume $b_m = 0, m = 1, 2, 6, 7, 8$ and $d_m = 0, m = 1, 2, 3, 4, 5$. Write out the matrix A for these values. For the matrix, explore the effect of growth if you have a computer to use, otherwise use the following: for the matrix I find there is the largest eigenvalue is 1.0016, with the following associated eigenvector:

$$\mathbf{N}_e = \begin{pmatrix} 0.1757 \\ 0.3459 \\ 0.3404 \\ 0.3350 \\ 0.3297 \\ 0.3137 \\ 0.2891 \\ 0.5750 \end{pmatrix}.$$

How does this growth rate compare with the US population growth between 1990 and 2000? What can you say about the predicted equilibrium age distribution compared to that projected for 2010?

3. Consider the second-order logistic difference equation

$$x_{m+1} = r x_m (1 - x_{m-1}).$$

(a) By exmaining the linear stability of the equilibrium $x_e = 1 - 1/r$, show that for $1 < r < 2$ the equilibrium is stable, and that the decay factor is complex when $r > 1.25$
(b) Show that for $r > 2$ the equilibrium is unstable.