A MINI COURSE ON SCALAR CURVATURE
LECTURE NOTES FOR SCGAS 2025

CHAO LI

Abstract. These are my lecture notes for Southern California Geometric Analysis Seminar, Feb 2025. I focus on Riemannian manifolds with positive scalar curvature, especially its connection to geometric variational problems.

Throughout these notes, we adopt the following conventions.

- Unless otherwise indicated, manifolds and Riemannian metrics are smooth.
- Curvature tensors are defined as follows.

$$R(X,Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z, \quad R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$$

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y) = \mathrm{tr}\, R(\cdot, X, Y, \cdot), \quad R = \mathrm{tr}\,\mathrm{Ric}(\cdot,\cdot).$$

Contents

1. Motivations and basic examples

We start our discussion with some scenarios where scalar curvature plays an important role. This section is partly motivated by Rick Schoen's Nachdiplom Lectures in ETH-FIM, 2017.

1.1. **Einstein equations of general relativity.**

Suppose (S^{n+1}, g) is a Minkowski manifold. Einstein's theory of general relativity states that (S^{n+1}, g) represents a space time if it satisfies the Einstein equation:

$$\mathrm{Ric}_S - \frac{1}{2} R_S g = T. \tag{EE}$$

Here T is a symmetric $(0,2)$ tensor representing matter field in space time. A natural condition we assume for T is the *dominant energy condition* (DEC):

For any future time-like or null vector V, $-T(V, \cdot)^{\#}$ is also future time-like or null.

This is equivalent to state that mass-energy can never be observed to flow faster than light. In an orthonormal frame $\{e_j\}_{j=0}^n$ near a point $p \in S$ with e_0 a unit time-like vector, (DEC) requires that

$$T_{00} \geq \sqrt{\sum_{j=1}^n T_{0j}^2}.$$

Here $T_{0j} = T(e_0, e_j)$.

The Einstein equation (EE) is hyperbolic. To realize it as an initial value problem, let us consider $M^n \subset S^{n+1}$ a space-like hypersurface (i.e. $g|_M > 0$). Denote by II the second fundamental form of the embedding $M \hookrightarrow S$. The pair $(M, g|_M, \mathrm{II})$ is called an *initial data set*. One should think of $g|_M$ the initial value of g on M and II the initial derivative of g in the time direction.

Let us now slightly diverge and fix the convention of II. For a choice of unit normal vector field ν of the embedding of a hypersurface, define the (scalar) second fundamental form II by setting

$$-\mathrm{II}(X, Y)\nu = \nabla_X Y - \nabla_X^T Y, \quad \forall X, Y \in \Gamma(TM).$$

Note that with this convention, the unit sphere $S^n \hookrightarrow \mathbf{R}^{n+1}$, equipped with the outward unit normal ν, satisfies that $\mathrm{II}(X, Y) = \langle X, Y \rangle$. In general, II is a symmetric $(0,2)$ tensor, and its eigenvalues (all real) are called principal curvatures. We then set $H = \mathrm{tr}\, \mathrm{II}$ the (scalar) mean curvature of the embedding with respect to ν, and $\vec{H} = -H\nu$ the mean curvature vector (note that H depends on the choice of ν and $\vec{H}$ does not). The Gauss equation states that for an embedding $M^n \to S^{n+1}$,

$$R_S(X,Y,Z,W) = R_M(X,Y,Z,W) + g(\mathrm{II}(X,Z)\nu, \mathrm{II}(Y,W)\nu) - g(\mathrm{II}(X,W)\nu, \mathrm{II}(Y,Z)\nu).$$

Back to the discussion on (EE). In a local frame $\{e_j\}_{j=0}^n$ with e_0 a unit normal of M, restricting (EE) to (e_0, e_0) gives (recall that $g(e_0,e_0) = -1$):

$$\mathrm{Ric}_S(e_0,e_0) + \frac{1}{2}R_S = T_{00}.$$

Expanding the scalar curvature term, we have (again $g(e_0,e_0) = -1$):

$$\frac{1}{2}R_S = -\sum_{j=1}^n R_{0jj0} + \sum_{1\le i<j\le n} R_{ijji} = -\mathrm{Ric}_S(e_0,e_0) + \sum_{1\le i<j\le n} R_{ijji}.$$

By the Gauss equation, we have:

$$R^S_{ijji} = R^M_{ijji} + \mathrm{II}_{ii}\,\mathrm{II}_{jj} - \mathrm{II}_{ij}^2\,.$$

And hence $\frac{1}{2}R_S = -\mathrm{Ric}_S(e_0,e_0) + \frac{1}{2}R_M + \frac{1}{2}(\mathrm{tr}\,\mathrm{II})^2 - \frac{1}{2}|\,\mathrm{II}\,|^2$. Plugging this into (EE), we obtain:

$$R_M + (\mathrm{tr}\,\mathrm{II})^2 - |\,\mathrm{II}\,|^2 = R_S + 2\,\mathrm{Ric}_S(e_0,e_0) = 2T_{00}. \tag{1.1}$$

A similar computation via evaluating (EE) in (e_0, e_j) and the Codazzi equation yields

$$\mathrm{div}_g(\mathrm{II} - (\mathrm{tr}_g\,\mathrm{II})g) = J := T(e_0, \cdot). \tag{1.2}$$

Together, (1.1) and (1.2) are called the Einstein constraint equations for $(M^n, g|_M, \mathrm{II})$.

A particularly important case is when $\mathrm{II} = 0$ - in this case, $(M^n, g|_M)$ is called time symmetric. Note that (1.2) is automatic, and the dominant energy condition implies that $T_{00} \ge 0$. Thus, the Einstein constraint equations implies that

$$R_M \ge 0.$$

Thus, manifolds with positive (or nonnegative) scalar curvature naturally arises in mathematical general relativity.

1.2. **Variation of total scalar curvature.** Fix $n \ge 3$. For a closed manifold M^n, define the following Einstein-Hilbert functional on a Riemannian metric g:

$$\mathscr{R}(g) = \int_M R_g dV_g.$$

Let us find when g is a critical point of $\mathscr{R}$ among volume-preserving deformations. Let h be a compactly supported symmetric $(0,2)$ tensor. For small t, define $g_t = g + th$, and $\bar{g}_t = \mathrm{vol}(g_t)^{-\frac{2}{n}} g_t$. In local coordinates, recall that

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}),$$

$$\mathrm{Ric}_{ij} = \sum_k (\Gamma^k_{ij,k} - \Gamma^k_{ki,j} + \sum_l (\Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{jl}\Gamma^l_{ki})).$$

Without loss of generality, assume $\{x^i\}$ is normal at a point p. We compute the derivatives of geometric quantities defined with g_t (a dot means taking derivative with respect to t) at p:

$$\dot{\Gamma}^k_{ij} = \frac{1}{2}\sum_l g^{kl}(h_{il,j} + h_{jl,i} - h_{ij,l}),$$

$$\dot{\mathrm{Ric}}_{ij} = \sum_k (\dot{\Gamma}^k_{ij,k} - \dot{\Gamma}^k_{ki,j}).$$

Use $R = \sum g^{ij}\,\mathrm{Ric}_{ij}$ and $(g^{-1})' = -g^{-1}\dot{g}g^{-1}$ (particularly $(g^{-1})'_{ij} = -h_{ij}$ at p), we find

$$\dot{R} = \sum_{i,j}\left(-h_{ij}\,\mathrm{Ric}_{ij} + \sum_k g^{ij}(\dot{\Gamma}^k_{ij,k} - \dot{\Gamma}^k_{ki,j})\right) = -\langle h, \mathrm{Ric}\rangle + \sum_{i,k}\dot{\Gamma}^k_{ii,k} - \sum_{i,k}\Gamma^k_{ki,i}.$$

Observe that the second and third summand are both divergence terms[1], which integrates to zero by the divergence theorem. Also, we have that $\frac{d}{dt}dV_{g_t} = \frac{1}{2}\,\mathrm{tr}_{g_t}(h)dV_{g_t}$. Hence we have

$$\frac{d}{dt}\mathscr{R}(g_t) = -\int_M \left\langle h, \mathrm{Ric}_{g_t} - \frac{1}{2}R_t g_t\right\rangle dV_{g_t}.$$

Thus, we normalize with volume and obtain that

$$\left.\frac{d}{dt}\right|_{t=0}\mathscr{R}(\bar{g}_t) = -\,\mathrm{vol}(g)^{\frac{2-n}{n}}\int_M \left\langle h, \mathrm{Ric}_g - \frac{1}{2}R_g g + \frac{n-2}{n}\mathscr{R}(g)g\right\rangle dV_g.$$

Proposition 1.1. *If g is a critical point for $\mathscr{R}$ among volume preserving deformations, then $\mathrm{Ric}_g = cg$ for some constant c. In other words, such critical points are Einstein metrics.*

However, as we shall see very soon, the functional $\mathscr{R}$ is unbounded among all metrics with unit volume.

1.3. **Conformal deformations.** Instead, we consider $\mathscr{R}(g)$ for g among a smaller class of metrics. Take g_0 a metric on M^n, $n \geq 3$.

Definition 1.2. Define the conformal class of g_0 as

$$[g_0] = \{g = e^{2u}g_0 : u \in C^\infty(M)\},$$

and the Yamabe invariant of this conformal class as

$$Y([g_0]) = \inf\{\mathscr{R}(g) : g \in [g_0], \mathrm{vol}(g) = 1\}.$$

[1] In fact, a careful computation gives

$$\dot{R} = -\langle h, \mathrm{Ric}\rangle + \mathrm{div}_g\,\mathrm{div}_g\,h - \Delta(\mathrm{tr}_g\,h).$$

Lemma 1.3. *The scalar curvature of* $g = u^{\frac{4}{n-2}} g_0$ *is given by*

$$R_g = -c(n)^{-1} u^{\frac{n+2}{n-2}} Lu,$$

where $c(n) = \frac{n-2}{4(n-1)}$, $L = \Delta_{g_0} - c(n)R_{g_0}$ *is called the conformal Laplacian.*

Proof. Set $f = \frac{2}{n-2}\log u$ so $g = e^{2f} g_0$. Take normal coordinates around a point p. Use the expression of Christoffel symbols, we may relate Γ and Γ_0:

$$\Gamma_{ii}^k = (\Gamma_0)_{ij}^k + \delta_i^k \partial_j f + \delta_j^k \partial_i f - \delta_i^j \partial_k f.$$

Putting these into the formula for Ricci curvature and obtain:

$$\mathrm{Ric}_{ij} = (\mathrm{Ric}_0)_{ij} - (n-2)[\partial_i\partial_j f - (\partial_i f)(\partial_j f)] - (\Delta_{g_0} f + (n-2)|\nabla_{g_0} f|^2)(g_0)_{ij}.$$

Take trace and obtain that

$$R_g = e^{-2f}(R_{g_0} - 2(n-1)\Delta_{g_0} f - 2(n-1)(n-2)|\nabla_{g_0} f|^2).$$

Replace $f = \frac{2}{n-2}\log u$ and obtain the desired formula. □

We make two remarks here. First, since $-L$ is self-adjoint, it has real eigenvalues. Second, note that $dV_g = u^{\frac{2n}{n-2}} dV_{g_0}$. Thus, we conclude that

$$\mathscr{R}(g) = c(n)^{-1} \int_M \left(|\nabla_{g_0} u|^2 + c(n)R_{g_0} u^2\right) dV_{g_0}.$$

Therefore, by the Sobolev inequality (note $2^* = \frac{2n}{n-2}$), we conclude that

$$Y([g_0]) = \inf\left\{c(n)^{-1}\int_M |\nabla_{g_0} u|^2 + c(n)R_{g_0} u^2 : \int_M u^{\frac{2n}{n-2}} = 1\right\}.$$

exists.

It is natural to ask whether $Y([g_0])$ is achieved by $u \in C^\infty(M)$. This is called the Yamabe problem. By a simple computation, if $u \in C^\infty(M)$ achieves $Y([g_0])$, then $g = u^{\frac{4}{n-2}} g_0$ has constant scalar curvature. It is completely resolved by the combined work of Yamabe, Trudinger, Aubin and Schoen.

Let us instead focus on a simpler yet important conformal invariant: the sign of $Y([g_0])$. We have the following theorem.

Theorem 1.4. *Let* $n \geq 3$, (M^n, g_0) *be a closed Riemannian manifold. Then the conformal class* $[g_0]$ *belongs to exactly one of the following three cases:*

(1) $Y([g_0]) > 0 \Leftrightarrow$ *there exists* $g \in [g_0], R_g > 0$ *everywhere* $\Leftrightarrow \lambda_1(-L) > 0$.
(2) $Y([g_0]) = 0 \Leftrightarrow$ *there exists* $g \in [g_0], R_g = 0$ *everywhere* $\Leftrightarrow \lambda_1(-L) = 0$.
(3) $Y([g_0]) < 0 \Leftrightarrow$ *there exists* $g \in [g_0], R_g < 0$ *everywhere* $\Leftrightarrow \lambda_1(-L) < 0$.

Here $\lambda_1(-L)$ *is the first eigenvalue of* $-L$.

Proof. This is a direct consequence of the variational characterization of λ_1:

$$\lambda_1(-L) = \inf_{u \in C^1(M)} \frac{\int_M |\nabla u|^2 + c(n)R_{g_0} u^2}{\int_M u^2},$$

and the fact that the first eigenfunction is positive everywhere. ☐

Corollary 1.5. *Let (M^n, g_0) be a closed Riemannian manifold with $n \geq 3$. If $\mathscr{R}(g_0) < 0$ then $Y([g_0]) < 0$.*

Proof. Taking $u = 1$ into the variational characterization of $\lambda_1(-L)$, we find that $\lambda_1(-L) < 0$. ☐

1.4. **Basic constructions, the trichotomy theorem.** We review some basic constructions of Riemannian manifolds related to scalar curvature.

1.4.1. *Warped product.* Let us examine two basic warped products of manifolds.

Lemma 1.6. *Given (M^n, g) and $u \in C^\infty(M)$, $u > 0$. The warped product $(M^n \times [-1,1], \tilde{g} = g + u^2 dt^2)$ has scalar curvature*

$$R_{\tilde{g}} = R_g - \frac{2\Delta_g u}{u}. \tag{1.3}$$

Proof. At a point on $M \times [-1,1]$, take coordinates $\{x^j\}_{j=1}^n$ normal for M. We have that

$$\tilde{\nabla}_{\partial_i} \partial_t = \frac{u_i}{u} \partial_t, \quad \tilde{\nabla}_{\partial_t} \partial_t = -\sum_i u_i u \partial_i.$$

Since $\partial_1, \cdots, \partial_n, \frac{\partial_t}{u}$ is orthonormal, we have that

$$\begin{aligned} R_{\tilde{g}} &= R_g + 2 \sum_i \left(\langle \tilde{\nabla}^2_{\partial_i, \partial_t/u} (\partial_t/u), \partial_i - \tilde{\nabla}^2_{(\partial_t/u), \partial_i} \partial_t/u, \partial_i \right) \\ &= R_g - \frac{2\Delta_g u}{u}. \end{aligned}$$

☐

(1.3) will be used later in comparison with the stability inequality for minimal hypersurfaces.

Lemma 1.7. *Givem (M^n, g) and $u \in C^\infty([-1,1])$, $u > 0$. The warped product $(M^n \times [-1,1], \tilde{g} = u^2 g + dt^2)$ has scalar curvature*

$$R_{\tilde{g}} = u^{-2} R_g - 2n \frac{u''}{u} - n(n-1) \frac{(u')^2}{u^2}. \tag{1.4}$$

Proof. We provide here a proof which originates from the variation of surface area. Denote by $\Sigma = M \times \{t_0\}$. Along Σ, ∂_t is a unit normal vector field. Let II be the second fundamental form of $\Sigma \hookrightarrow M \times [-1,1]$, and take $\{e_i\}_{i=1}^n$ an orthonormal frame locally on Σ.

At a point on Σ, on one hand, by the Gauss equation, we have that

$$\begin{aligned} R_{\tilde g} - 2\,\mathrm{Ric}_{\tilde g}(\partial_t, \partial_t) &= \sum_{i,j=1}^{n} R^M_{ijji} \\ &= \sum_{i,j=1}^{n-1} (R^\Sigma_{ijji} - \mathrm{II}_{ii}\,\mathrm{II}_{jj} + \mathrm{II}^2_{ij}) \\ &= R_\Sigma - H^2 + |\,\mathrm{II}\,|^2. \end{aligned}$$

Here we used R_Σ to denote the scalar curvature of Σ with the induced metric. On the other hand, the vector field ∂_t is pushing Σ with unit speed in its normal direction. Using the second variation formula of area (we will see a more general version of this later), we conclude that

$$\frac{\partial H}{\partial t} = -\,\mathrm{Ric}_{\tilde g}(\partial_t, \partial_t) - |\,\mathrm{II}\,|^2.$$

Putting these to cancel the $\mathrm{Ric}_{\tilde g}$ term, we have:

$$R_{\tilde g} = R_\Sigma - (H^2 + |\,\mathrm{II}\,|^2) - 2\frac{\partial H}{\partial t}. \tag{1.5}$$

Now we compute II and H with the tube formula [Gra04] (see also [Gro91, p. 39]). Indeed, we have that $\mathrm{II} = \frac{1}{2}\frac{d}{dt} g|_{\Sigma_t}$, where Σ_t is equi-distant hypersurfaces moving in the ∂_t direction. Therefore, we have that

$$\mathrm{II} = u u_t g, \quad H = \mathrm{tr}_{u^2 g}\,\mathrm{II} = n u_t u^{-1}, \quad |\,\mathrm{II}\,|^2 = n\frac{(u')^2}{u^2}, \quad \frac{\partial H}{\partial t} = n\left(\frac{u'}{u}\right)'.$$

(1.4) is obtained by plugging these into 1.5. □

1.4.2. *Surgery.* We recall the notion of surgery from topology. Given a manifold M^n and an embedded $S^p \times D^q \subset M$ with $p + q = n$, since

$$\partial(S^p \times D^q) = S^p \times S^{q-1} = \partial(D^{q+1} \times S^{q-1}),$$

we may remove $S^p \times D^q$ and glue in $D^{p+1} \times S^{q-1}$ along their common boundary $S^p \times S^{q-1}$[2], obtaining a new manifold M'. We call p the dimension and q the codimension of the surgery.

Theorem 1.8 (Schoen-Yau [SY79b], Gromov-Lawson [GL80])**.** *If (M, g) satisfies that $R_g > 0$ and M' is obtained from M by a codimension at least 3 surgery, then M' admits a positive scalar curvature metric g'.*

In fact, one may perform a surgery in a purely local fashing: the metric g' can be chosen to equal g away from the surgery region, and its scalar curvature decreases by an arbitrarily small amount within it.

[2]One should be careful that different gluing diffeomorphisms along $S^p \times S^{q-1}$ may result in manifolds that are not diffeomorphic.

A particularly important case is when $n \geq 3$ and $p = 0$. Performing a 0-surgery on $p_j \in M_j$, $j = 1, 2$ gives the connected sum of M_1 and M_2, denoted by $M_1 \# M_2$. Taking connected sums is a simple way to construct new manifolds with scalar curvature lower bounds from existing ones.

1.4.3. *The trichotomy theorem.* We will use the above constructions to prove the following.

Theorem 1.9. *Every closed manifold M^n, $n \geq 3$, admits a Riemannian metric with negative scalar curvature.*

By Corollary 1.5, it suffices to construct a metric with total negative scalar curvature. We divide the construction into several steps.

We first observe that it suffices to construct a metric g on S^n, $n \geq 3$, such that $\mathscr{R}(g) < -1$. Indeed, for any Riemannian manifold (M^n, g_0), choose $\lambda > 0$ sufficiently small such that $\mathscr{R}(\lambda^2 g) = \lambda^{n-2}\mathscr{R}(g) < -|\mathscr{R}(g_0)|$. By Theorem 1.8, for every $\delta > 0$, there exists a metric $\tilde{g}$ on $M^n \# S^n$, with sufficiently small surgery region, such that $|\mathscr{R}(\tilde{g}) - \mathscr{R}(g_0) - \mathscr{R}(\lambda^2 g)| < \delta$. Therefore the metric $\tilde{g}$ has negative total scalar curvature.

We now focus on the construction of g, which builds upon (1.5).

Proposition 1.10. *Suppose (M_i^n, g_i), $i = 1, 2$, are compact manifolds with isometric boundary Σ. For each $\varepsilon > 0$, there exists a smooth metric g_ε on $M = M_1 \sqcup_\Sigma M_2$, $g = g_j$ away from the ε-neighborhood of M_j, and*

$$\left| \int_M R_{g_\varepsilon} dV - \int_{M_1} R_{g_1} dV - \int_{M_2} R_{g_2} dV - 2\int_\Sigma (H_1 + H_2) dA \right| < \varepsilon. \tag{1.6}$$

Here H_1, H_2 are the mean curvature of Σ embedded in M_1, M_2, respectively, taken with respect to the outward unit normal vector field.

Sketch of proof. Set $M = M_1 \sqcup_\Sigma M_2$, where we identify ∂M_1 and ∂M_2 by the isometry. M is a smooth manifold with Σ embedds into it. On M, define a metric g such that $g = g_j$ in $M_j \subset M$. Then g is smooth up to Σ from both sides, and is only Lipschitz along Σ.

Take Fermi coordinates on both sides of Σ, and let t be the signed distance function from Σ such that ∂_t points into M_2. The mean curvature of t-level sets are well defined and may be discontinuous along Σ. Still, (1.5) implies that:

$$R_M = R_\Sigma - (H^2 + |\,\mathrm{II}\,|^2) - 2\frac{\partial H}{\partial t}.$$

Note that the RHS of this expression is bounded even along Σ, except possibly for the last term. On the other hand, the last term has a distribution along Σ, which equals to $-H_2 - H_1$ (note that H_2 is taken with respect to $-\partial_t$). (1.6) is formally obtained by integrating this expression.

□

Remark 1.11. In [Mia02], Miao carried out this smoothing rigorously. Precisely, he computed the scalar curvature, using (1.5), for a fiber wise mollification of the metrics g_t of equi-distant hypersurfaces from Σ.

Remark 1.12. Gluing/smoothing construction of scalar curvature has been extensively investigated, see, for instance, [BMN11, BH23]. In particular, if $H_1 + H_2 > 0$ holds along Σ, then there exists a smooth of g which preserves pointwise scalar curvature lower bounds.

To finish the proof of Theorem 1.9, we write $S^n = S^n_+ \sqcup S^n_-$. Pick $p \in S^{n-1} = \partial S^n_+ = \partial S^n_-$. Locally near p, S^{n-1} is C^1 close to $\mathbf{R}^{n-1}$.

Since $n \geq 3$, one may attach a sequence of spheres (S^{n-1}) with small radius inside S^n_+ near p, such that $\int_{S^{n-1}} H dA < -100$. Denote the outcome by (S^n_+, g_1). Apply the same construction and obtain (S^n_-, g_2) such that $g_1|_{S^{n-1}} = g_2|_{S^{n-1}}$. Apply Proposition 1.10, since $\int_{S^{n-1}}(H_1+H_2) < -200$, we may smooth the metric and obtain a smooth metric on S^n with negative total scalar curvature.

Using a similar idea, one may prove that $\mathscr{R}(g)$ is unbounded among metrics with unit volume. We leave this as an exercise.

In [KW75], Kazdan-Warner proved the following trichotomy for Riemannian manifolds.

Theorem 1.13. *Let $n \geq 3$. For a closed manifold M^n, exactly one of the following three statements hold on M:*

1. *Every $f \in C^\infty(M)$ can be realized as the scalar curvature function of a Riemannian metric g.*
2. *$f \in C^\infty(M)$ can be realized as the scalar curvature function of a Riemannian metric if and only if $f < 0$ somewhere or $f = 0$ everywhere.*
3. *$f \in C^\infty(M)$ can be realized as the scalar curvature function of a Riemannian metric if and only if $f < 0$ somewhere.*

In fact, it was proved by Lohkamp [Loh95] that scalar curvature (even Ricci curvature!) satisfies a h-principle: one may locally arbitrarily decrease scalar curvature of any metric with a small C^0 perturbation. It is thus concluded that having (somewhere) negative scalar curvature does not put any topological condition on a manifold. However, from Theorem 1.13, only manifolds in class (1) admits a metric with positive scalar curvature (PSC); manifolds in class (2) admits a metric g with $R_g = 0$, but no PSC metric.

Example 1.14. *K3 surface.* $M = \{\sum_{j=1}^4 x_j^4 = 0 : [x_1, x_2, x_3, x_4] \in \mathbf{C}P^3\}$ *admits a Ricci flat (hence scalar flat) metric by the Calabi-Yau theorem. However, it does not admit any PSC metric.*

Futaki [Fut93] proved that if a simply connected manifold of dimensions at least 5 is in class (2), then it is the product of manifolds with special holonomy. On the other hand, closed simply connected manifolds with holonomy in $G2$ or $SU(4k+3)$ do admit PSC metrics, see [DWW05].

2. The obstruction problem, Schoen-Yau descent

Problem 2.1 (The obstruction problem for PSC)**.** Determine which smooth closed manifold M^n admits a Riemannian metric with positive scalar curvature.

Problem 2.1 has been a central topic in geometric analysis. Numerous tools have been developed for its investigation:

(1) Spinors. Lichnerowicz formula: for a section of the spinor bundle, one has
$$D^2 = \nabla^*\nabla + \frac{1}{4}R_g.$$
Here D is the Dirac operator. Thus, if $R_g > 0$ on a closed manifold, $\ker D = \{0\}$. This implies that $\hat{A}(M) = 0$ via the Atiyah-Singer index theorem. This approach has far-reaching consequences on the topology of PSC manifolds.

Theorem 2.2 ([GL80],[Sto92])**.** *Let $n \geq 5$ and M^n is a closed manifold with $\pi_1(M) = 0$. Then M admits a PSC metric if and only if:*
(a) *M is not spin,*
(b) *or M is spin and $\alpha(M) = 0$.*
Here $\alpha(M)$ is the α invariant of M.

(2) Minimal hypersurfaces. This is the focus of the remaining lectures.
(3) Ricci flow and the inverse mean curvature flow (especially in 3 dimensions).
(4) Level sets of harmonic functions (currently for 3-manifolds).
(5) Seiberg-Witten invariants for 4-manifolds.

Example 2.3. *K3 surface. For a closed 4-manifold M, $\hat{A}(M) = -\frac{1}{8}\sigma(M)$, here $\sigma(M)$ denotes the signature. Thus, $\hat{A}(K3) = -2$.*

A crucial example for the obstruction problem was the following:

Conjecture 2.4 (Geroch conjecture)**.** For all $n \geq 2$, the n-dimensional torus T^n admits no PSC Riemannian metric.

Conjecture 2.4 was proved in the affirmative by Schoen-Yau (at least when $n \leq 7$) and independently by Gromov-Lawson. The two proofs are entirely different, and have both motivated exciting developments. We will focus on the Schoen-Yau proof using minimal hypersurfaces. We will use the following result from geometric measure theory.

Theorem 2.5. *For $n \leq 7$, suppose (M^n, g) is a closed oriented Riemannian manifold. For any $\alpha \in H_{n-1}(M, \mathbf{Z})$, there exists an area minimizing representative*
$$\alpha = [\Sigma_1] + \cdots + [\Sigma_k].$$
In particular, $\{\Sigma_j\}_{j=1}^k$ is a disjoint union of embedded two-sided area minimizing hypersurface.

2.1. **First and second variation of minimal hypersurfaces.** Given a two-sided immersion $\Sigma^{n-1} \to M^n$, suppose $F : \Sigma^{n-1} \times (-\varepsilon, \varepsilon) \to M^n$ is a variation in the sense that:

(1) $F(\cdot, t) : \Sigma \to M$ is an immersion;
(2) $F(\cdot, 0) = \mathrm{id}$;
(3) $F(\cdot, t) = \mathrm{id}$ outside a compact subset of Σ.

Without loss of generality, let us assume that F is a normal variation, that is, $\frac{\partial}{\partial_t} F_t = f_t \nu_t$, where ν_t is a choice of unit normal vector field on $\Sigma_t := \operatorname{im} F(\cdot, t)$.

Theorem 2.6. *For a variation F with $\frac{\partial}{\partial_t}|_{t=0} F_t = f \in C_0^\infty(\Sigma)$, we have that:*

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}(\Sigma_t) = \int_\Sigma Hf,$$
$$\left.\frac{d^2}{dt^2}\right|_{t=0} \operatorname{vol}(\Sigma_t) = \int_\Sigma |\nabla f|^2 - (|\operatorname{II}|^2 + \operatorname{Ric}(\nu, \nu)) f^2 + H^2 f^2 + H\dot{f}.$$

In fact, both formulas have pointwise version as follows:

$$\frac{d}{dt} dV_{\Sigma_t} = H_t f_t dV_{\Sigma_t},$$
$$\frac{d}{dt} H = -\Delta f - (\operatorname{Ric}(\nu, \nu) + |\operatorname{II}|^2) f.$$

On Σ, we call $J = -\Delta - (\operatorname{Ric}(\nu, \nu) + |\operatorname{II}|^2)$ the Jacobi operator. J is elliptic and self-adjoint.

Definition 2.7. Call a two-sided immersion $\Sigma^{n-1} \to (M^n, g)$ minimal, if $H = 0$ along Σ. Thus, Σ is minimal if and only if $\frac{d}{dt}|_{t=0} \operatorname{vol}(\Sigma_t) = 0$ for all variations.

Call a minimal immersion $\Sigma \to (M^n, g)$ stable, if $\frac{d^2}{dt^2}|_{t=0} \operatorname{vol}(\Sigma_t) \geq 0$ for all variations. When Σ is two-sided, stability is equivalent to

$$\int_\Sigma |\nabla f|^2 - (\operatorname{Ric}(\nu, \nu) + |\operatorname{II}|^2) f^2 \geq 0, \quad \forall f \in C_0^\infty(\Sigma).$$

By the variational characterization of the first eigenvalue, a two-sided minimal immersion $\Sigma^{n-1} \to (M^n, g)$ is stable if and only if $\lambda_1(J) \geq 0$.

2.2. **Proof of Geroch when $n = 3$.** Assume (T^3, g) has $R_g > 0$. Since take generators $\{dx^1, dx^2, dx^3\}$ of $H^2(T^3, \mathbf{Z}) \simeq \mathbf{Z}^3$. Consider the minimization problem

$$\inf \left\{ \operatorname{area}(\Sigma) : \Sigma^2 \subset T^3, \int_\Sigma dx^1 \wedge dx^2 = 1 \right\}.$$

Since the integration defines an integral homology class in $H_2(M)$, we fine an embedded two-sided stable minimal surface $\Sigma = \Sigma_1 + \cdots + \Sigma_k$. Therefore, on each Σ_j, we have that

$$\int_{\Sigma_j} |\nabla f|^2 - (\operatorname{Ric}_g(\nu,\nu) + |\operatorname{II}|^2) f^2 \geq 0, \quad \forall f \in C^\infty(\Sigma_j).$$

We derive a contradiction as follows. On one hand, we claim that $H^1(\Sigma, \mathbf{R}) \neq 0$. To see this, set $\omega_j = [dx^j|_\Sigma] \in H^1_{dR}(\Sigma, \mathbf{R})$, $j = 1, 2$. Then we have that $\omega_j \neq 0$. Otherwise, if $\omega_1 = df$, then we have that

$$1 = \int_\Sigma \omega_1 \wedge \omega_2 = \int_\Sigma df \wedge \omega_2 = \int_\Sigma d(f \wedge \omega_2) - f \wedge d\omega_2 = 0.$$

On the other hand, fix $j \in \{1, \cdots, k\}$. We use the Gauss equation to rewrite the curvature terms in the stability inequality as follows. Take a local orthonormal frame on Σ with $e_n = \nu$. Then

$$\begin{aligned} R_M - 2\operatorname{Ric}_g(\nu,\nu) &= \sum_{i,j=1}^{n-1} R^M_{ijji} \\ &= \sum_{i,j=1}^{n-1} (R^\Sigma_{ijji} - \operatorname{II}_{ii}\operatorname{II}_{jj} + |\operatorname{II}_{ij}|^2) \\ &= R_\Sigma - H^2 + |\operatorname{II}|^2. \end{aligned}$$

Thus, we have that

$$\operatorname{Ric}_g(\nu,\nu) + |\operatorname{II}|^2 = \frac{1}{2}(R_M - R_\Sigma + |\operatorname{II}|^2 + H^2). \tag{2.1}$$

Therefore, the stability inequality implies that

$$\int_{\Sigma_j} |\nabla f|^2 + \frac{1}{2} R_{\Sigma_j} f^2 \geq \int_{\Sigma_j} \frac{1}{2}(R_M + |\operatorname{II}|^2) f^2 > 0, \quad \forall f \in C^\infty(\Sigma_j).$$

Take $f = 1$ above. Note that Σ_j is a 2-dimensional surface, so $\frac{1}{2}R_{\Sigma_j} = K_{\Sigma_j}$. Thus, the Gauss-Bonnet theorem implies that

$$2\pi\chi(\Sigma_j) = \int_{\Sigma_j} K_{\Sigma_j} > 0,$$

and hence Σ_j is diffeomorphic to S^2. Therefore, Σ is the disjoint union of two-spheres, and thus does not support any nontrivial class in H^1, contradiction.

2.3. **Schoen-Yau descent, minimal slicing.** Inductive descent argument: construct a nested family of oriented submanifolds

$$\Sigma_k \subset \Sigma_{k+1} \subset \cdots \subset \Sigma_n = (M^n, g),$$

such that $\dim \Sigma_k = k$. Assuming $R_g > 0$ on M, we would also like to construct a PSC metric on each Σ_k. Such a nested family is called a k-slicing. The existence of a k-slicing is usually guaranteed by topological assumptions, particularly that the homology of M^n is sufficiently large.

Example 2.8. *A trivial example of a k-slicing of minimal submanifolds can be constructed in $X^k \times T^{n-k}$, equipped with a product metric $g + g_0$, where g_0 is the flat product metric on T^{n-k}. In this case, we may take the nested family of totally geodesic embeddings*

$$X \subset X \times S^1 \subset \cdots \subset X \times T^{n-k}.$$

We now describe two approaches to carry out the inductive descent argument.

2.3.1. *Conformal descent.* The first approach, called the conformal descent argument, utilizes the connection between Jacobi operator of minimal hypersurfaces and the conformal Laplacian.

Proposition 2.9. *Suppose $\Sigma^{n-1} \subset (M^n, g)$ is a two-sided stable minimal hypersurface, and $R_g > 0$. Then the induced metric on Σ is Yabame positive - that is, it has pointwise positive scalar curvature after a conformal change.*

Proof. We write the stability inequality on a minimal hypersurface using the Schoen-Yau rearrangement (2.1): for all $f \in C_0^\infty(\Sigma)$, we have

$$\int_\Sigma |\nabla f|^2 - \frac{1}{2}(R_M - R_\Sigma + |\,\mathrm{II}\,|^2) f^2 \geq 0 \quad \Rightarrow \quad \int_\Sigma |\nabla f|^2 + \frac{1}{2} R_\Sigma f^2 > 0.$$

Recall that the conformal Laplacian on Σ is given by $L = -\Delta + c(n) R_\Sigma$ with $c(n) = \frac{n-3}{2(n-2)}$. Using the fact that $\frac{1}{2} > \frac{n-3}{2(n-2)}$, we have that for all $f \in C_0^\infty(\Sigma)$,

$$\int_\Sigma 2|\nabla f|^2 + R_\Sigma f^2 > 0 \quad \Rightarrow \quad \int_\Sigma \frac{2(n-2)}{n-3} |\nabla f|^2 + R_\Sigma f^2 > 0.$$

Thus, $\lambda_1(L) > 0$. By Theorem 1.4, we conclude that $[g|_\Sigma]$ is Yamabe positive. ☐

Therefore, if we may inductively construct $\Sigma_k \subset \Sigma_{k+1}$ as a stable minimal embedding, then

$$\Sigma_{k+1} \text{ is PSC } \Rightarrow \Sigma_k \text{ is PSC.}$$

Proposition 2.10. *For $2 \leq n \leq 7$, if M^n is a closed manifold with $\omega_1, \cdots, \omega_{n-1} \in H^1_{dR}(M, \mathbf{R})$ such that $\omega_1 \wedge \cdots \wedge \omega_{n-1} \neq 0 \in H^{n-1}_{dR}(M, \mathbf{R})$, then M does not admit a PSC metric.*

Proof. Induction on n. When $n = 2$, the only PSC 2-dimensional manifold is diffeomorphic to S^2, which does not have any nontrivial element in H^1_{dR}. For $3 \leq n \leq 7$, we use the de Rham theorem to find an integral homology class $\alpha \in H_{n-1}(M, \mathbf{Z})$ and an area minimizing hypersurface Σ_{n-1} representing α such that

$$\int_{\Sigma_{n-1}} \omega_1 \wedge \cdots \wedge \omega_{n-1} \neq 0.$$

With the same proof as before, we see that

$$\omega_1|_{\Sigma_{n-1}}, \cdots, \omega_{n-1}|_{\Sigma_{n-1}} \neq 0 \in H^1_{dR}(\Sigma_{n-1}, \mathbf{R}).$$

Also the above gives that $\omega_1|_{\Sigma_{n-1}} \wedge \cdots \wedge \omega_{n-2}|_{\Sigma_{n-1}} \neq 0 \in H^{n-2}_{dR}(\Sigma_{n-1}, \mathbf{R})$. This finishes the proof, since if M carries a PSC metric, then so does Σ_{n-1}, contradiction. □

2.3.2. *Warped product descent.* The second approach, called the warped product descent (also called S^1-symmetrization technique by Gromov), uses a connection between (1.3) and the stability inequality. This approach is more quantitative, as it preserves the scalar curvature lower bound in the descent.

Again, suppose that $\Sigma^{n-1} \subset (M^n, g)$ is a compact two-sided stable minimal hypersurface. The stability inequality and the Schoen-Yau rearrangement (2.1) implies that $\lambda_1(-\Delta - \frac{1}{2}(R_M - R_\Sigma + |\,\mathrm{II}\,|^2)) \geq 0$. Therefore, the first eigenfunction u of J satisfies that $u > 0$ and

$$-\Delta u + \frac{1}{2} R_\Sigma u = \frac{1}{2}(R_M + |\,\mathrm{II}\,|^2 + \lambda_1) u \tag{2.2}$$

on Σ.

Consider the warped product $(\Sigma \times S^1, \tilde{g} = g_\Sigma + u^2 dt^2)$. By (1.3), we have that

$$\begin{aligned} R(\tilde{g}) &= R_\Sigma - 2u^{-1}\Delta u \\ &\geq R_\Sigma + 2u^{-1}\left(\frac{1}{2}(R_M - R_\Sigma + |\,\mathrm{II}\,|^2) + \lambda_1\right) u \\ &= R_M + \lambda_1. \end{aligned}$$

Hence the scalar curvature lower bound is preserved.

Next, we seek to minimize volume of $\Sigma_{n-2} \times S^1 \subset (\Sigma_{n-1} \times S^1, \tilde{g})$. That is:

$$\inf \left\{ \mathrm{vol}_{\tilde{g}}(\Sigma_{n-2} \times S^1) : \Sigma_{n-2} \subset \Sigma_{n-1} \text{ represents a homology class } \alpha \in H_{n-2}(\Sigma_{n-1}, \mathbf{Z}) \right\}.$$

Equivalently, we consider

$$\inf \left\{ \int_{\Sigma_{n-2}} u_{n-1} : \Sigma_{n-2} \subset \Sigma_{n-1} \text{ representing } \alpha \right\},$$

here u_{n-1} is the first eigenfunction of the Jacobi operator on Σ_{n-1}.

Inductively, suppose we have constructed Σ_{n-k+1}. We then minimize

$$\Sigma_{n-k} \times T^{k-1} \subset \left(\Sigma_{n-k+1} \times T^{k-1}, g_{n-k+1} + u^2_{n-k+1} dt^2_{n-k+1} + \cdots + u^2_{n-1} dt^2_{n-1}\right)$$

among hypersurfaces Σ_{n-k} in a suitable homology class of Σ_{n-k+1}, g_{n-k+1} is the induced metric of $\Sigma_{n-k+1} \subset M$. Equivalently, we may minimize the weighted volume

$$\int_{\Sigma_{n-k}} u_{n-k+1} \cdots u_{n-1}.$$

Finally, we choose u_{n-k} be the first eigenfunction of the Jacobi operator (with respect to the weighted volume functional) on Σ_{n-k}.

Let us carry this out in details when $n = 4$.

Setup: Let $\Gamma^3 \subset (M^4, g)$ be stable minimal, $u > 0$ be the first Jacobi eigenfunction on Γ, satisfying $-\Delta_\Gamma u - \frac{1}{2}(R_M - R_\Gamma + |\,\mathrm{II}\,|^2)u \geq 0$. Let $\Sigma^2 \subset \Gamma$ be stable for

$$\mathcal{A}(\Sigma) = \int_\Sigma u dA.$$

Our goal is to deduce geometric consequences on Σ.

For this, let's compute the first and second variation of $\mathcal{A}$. Let φ be the speed of a normal variation such that $\dot{\varphi} = 0$ (this can be arranged, for instance, by taking normal exponential maps with speed φ). Differentiating each term, we get:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(\Sigma_t) = \int_{\Sigma_t} \langle \nabla_\Gamma u, \nu \rangle \varphi + uH\varphi dA.$$

Hence on Σ we have that

$$H = -u^{-1} \langle \nabla_\Gamma u, \nu \rangle .$$

To differentiate this again to find the second variation, we note that $\dot{\nu} = -\nabla_\Sigma \varphi$. Indeed, denote by $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ the variation. Take local coordinates on x^i such that $dF(\partial_i)$ is normal at a point. $\partial_t \nu = \nabla_\Sigma \varphi$ follows from $\langle \partial_t \nu, \nu \rangle = 0$ and that

$$\langle \partial_t \nu, \partial_i F \rangle = -\langle \partial_t \partial_i F \rangle = -\langle \partial_i \partial_t F, \nu \rangle = -\langle \partial_i(\varphi\nu), \nu \rangle = -\partial_i \varphi.$$

Therefore, we find that

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{A}(\Sigma_t) = \int_\Sigma & \nabla^2_\Gamma u(\nu,\nu)\varphi^2 - \langle \nabla_\Gamma u, \nabla_\Sigma \varphi \rangle \varphi + \langle \nabla_\Gamma u, \nu \rangle H\varphi^2 \\ & - u(-\Delta_\Sigma u + (\mathrm{Ric}_\Gamma(\nu,\nu) + |\,\mathrm{II}_\Sigma\,|^2)\varphi)\varphi + uH^2\varphi^2.\end{aligned}$$

Next, we use the basic relation that (check this yourself)

$$\nabla^2_\Gamma u(\nu,\nu) = \Delta_\Gamma u - \Delta_\Sigma u + \langle \nabla_\Gamma u, \nu \rangle H.$$

Use the Schoen-Yau rearrangement trick to write $\mathrm{Ric}_\Gamma(\nu,\nu) + |\,\mathrm{II}_\Sigma\,|^2 = \frac{1}{2}(R_\Gamma - R_\Sigma + |\,\mathrm{II}_\Sigma\,|^2 + H^2)$. Rearranging terms and plugging in the expression for H, we have:

$$\begin{aligned}0 \leq \int_\Sigma & (\Delta_\Gamma u - \frac{1}{2}R_\Gamma u)\varphi^2 - \Delta_\Sigma u \varphi^2 - \frac{3}{2}u^{-1} \langle \nabla_\Gamma u, \nu \rangle^2 \varphi^2 \\ & - \langle \nabla_\Sigma u, \nabla_\Sigma \varphi \rangle \varphi - u\varphi\Delta_\Sigma\varphi + \frac{1}{2}uR_\Sigma\varphi^2 - \frac{1}{2}u|\,\mathrm{II}_\Sigma\,|^2\varphi^2.\end{aligned}$$

We throw away the last terms on each line above and. Use the assumption that $\Delta_\Gamma u - \frac{1}{2}R_\Gamma u \leq -\frac{1}{2}R_M u$ and integrate by parts (here all geometric operations are with respect to the induced metric on Σ):

$$\int_\Sigma \frac{1}{2} R_M u\varphi^2 \le \int_\Sigma -\Delta u\varphi^2 - \langle \nabla u, \nabla\varphi\rangle \varphi - u\varphi\Delta\varphi + \frac{1}{2} u R_\Sigma \varphi^2$$
$$= \int_\Sigma -\Delta u\varphi^2 + u|\nabla\varphi|^2 + uK_\Sigma\varphi^2.$$

Note that we have no further information on u, and hence we would like to cancel the terms involving u altogether. Set $\varphi = u^{-\frac{1}{2}}\psi$ and expand $\nabla\varphi = u^{-\frac{1}{2}}\nabla\psi - \frac{1}{2}u^{-\frac{3}{2}}\psi\nabla u$. Thus,

$$\int_\Sigma \frac{1}{2} R_M\psi^2 \le \int_\Sigma -\Delta u u^{-1}\psi^2 - u^{-1}\langle \nabla u, \nabla\psi\rangle \psi + \frac{1}{4}u^{-2}|\nabla u|^2\psi^2 + K_\Sigma\psi^2 + |\nabla\psi|^2$$
$$= \int_\Sigma -\frac{3}{4}u^{-2}|\nabla u|^2\psi^2 + u^{-1}\langle \nabla u, \nabla\psi\rangle \psi + |\nabla\psi|^2 + K_\Sigma\psi^2.$$

Finally, use AM-GM:

$$-\frac{3}{4}u^{-2}|\nabla u|^2\psi^2 + u^{-1}\langle \nabla u, \nabla\psi\rangle \psi - \frac{1}{3}|\nabla\psi|^2 \le 0.$$

Hence we conclude that

$$\int_\Sigma \frac{1}{2} R_M\psi^2 \le \int_\Sigma \frac{4}{3}|\nabla\psi^2| + K\psi^2, \quad \forall\psi \in C^\infty(\Sigma).$$

Plugging in $\psi = 1$ everywhere and using $R_M > 0$, we conclude that Σ is the disjoint union of two-spheres by the Gauss-Bonnet theorem. In general, we have that:

Theorem 2.11 (Schoen-Yau)**.** *Let $n \le 7$. Suppose (M^n, g) satisfies $R_g > 0$, and a weighted minimal k-slicing defined above exists. Then for each $k \le j \le n-1$, Σ_j is Yamabe positive. In particular, if $k = 2$, then Σ_2 is a union of two-spheres.*

By an argument completely analogous to the proof of Proposition 2.10, if M^n is a closed manifold with $\omega_1, \cdots, \omega_{n-k+1} \in H^1_{dR}(M^n, \mathbf{R})$ such that $\omega_1 \wedge \cdots \omega_{n-k+1} \neq 0$, then a minimal k-slicing exists.

Corollary 2.12. *Suppose $3 \le n \le 7$. if M^n is a closed manifold admitting a map of nonzero degree to T^n, then M^n does not carry a PSC Riemannian metric.*

Proof. Let $f : M \to T^n$ be a map with nonzero degree, and let $\omega_j = f^*(dx^j)$, $j = 1, \cdots, n$. Then

$$\int_M \omega_1 \wedge \cdots \wedge \omega_n = (\deg f) \int_{T^n} dx^1 \wedge \cdots \wedge dx^n \neq 0.$$

□

3. Geometric estimates, spectral extensions, μ-bubbles

Previously, we proved the following:

- A stable minimal $\Sigma^2 \subset (M^3, g)$ with $R_g \geq 1$ satisfies
$$\lambda_1(-\Delta_\Sigma + \frac{1}{2}R_\Sigma) \geq \frac{1}{2}.$$
- Suppose (M^3, g) satisfies $\lambda_1(-\Delta + \frac{1}{2}R_\Sigma) \geq \lambda > 0$ and let $u > 0$ be the first eigenfunction of $-\Delta + \frac{1}{2}R_M$. Then a weighted stable minimal surface Σ^2 (with weight u) satisfies
$$\lambda_1(-\frac{4}{3}\Delta_\Sigma + \frac{1}{2}R_\Sigma) \geq \lambda.$$

For more applications of (weighted) stable minimal surfaces, we would need to derive more precise geometric estimates on its size. We start witht the following simple observation.

Proposition 3.1. *Suppose (M^3, g) satisfies $R_g \geq 2$. Then any connected two-sided stable minimal surface $\Sigma^2 \subset M^3$ satisfies that* $\text{area}(\Sigma) \leq 4\pi$.

Proof. Plug $f = 1$ in
$$\int_\Sigma |\nabla f|^2 + K_\Sigma f^2 \geq \int_\Sigma f^2$$
(note that this follows from stability and taht $R_g \geq 2$). We have:
$$\text{area}(\Sigma) \leq 2\pi\chi(\Sigma).$$
Thus $\Sigma \simeq S^2$ and the above becomes $\text{area}(\Sigma) \leq 4\pi$. □

Remark 3.2. There is an interesting related rigidity result by Brendle-Bray-Neves: if $\pi_2(M) \neq 0$, then the least area homotopically nontrivial surface Σ must have area at most 4π. Moreover, if $\text{area}(\Sigma) = 4\pi$, then M is covered by $S^2(1) \times \mathbf{R}$.

3.1. Diameter estimates.

Recall the classical Bonnet's theorem.

Theorem 3.3 (Bonnet)**.** *Suppose that (Σ^2, g) has either empty or compact boundary, and satisfies $K_g \geq 1$. Then the length of any stable geodesic segment is $\leq \pi$. Consequently,*

(1) *If $\partial\Sigma = \emptyset$, then* $\text{diam}(\Sigma, g) \leq \pi$.
(2) *If $\partial\Sigma \neq \emptyset$, then* $\text{dist}(p, \partial\Sigma) \leq \pi$ *for all $p \in \Sigma$.*

An important observation due to Schoen-Yau states that one may replace the pointwise curvature condition by a spectral one and still obtain diameter estimates.

Theorem 3.4 (Schoen-Yau [SY83])**.** *Suppose that (Σ^2, g) has either empty or compact boundary, and satisfies that $\lambda_1(-\Delta + K) \geq 1$. Then:*

(1) *If $\partial\Sigma = \emptyset$ then* $\text{diam}(\Sigma, g) \leq \frac{2}{\sqrt{3}}\pi$.

(2) *If* $\partial\Sigma \neq \emptyset$ *then* $\operatorname{dist}(p, \partial\Sigma) \leq \frac{2}{\sqrt{3}}\pi$ *for all* $p \in \Sigma$.

Remark 3.5. In fact, for all $a \in (0, 4)$, one may replace the condition to $\lambda_1(-a\Delta + K) \geq 1$ and obtain analogous conclusions with upper bound $\frac{2}{\sqrt{4-a}}\pi$. The range $a < 4$ is sharp: the hyperbolic space satisfies that $\lambda_1(-\Delta) = \frac{1}{4}$ and hence $\lambda_1(-4\Delta + K) = 0$.

Proof. We give a proof that is closely related to the minimal slicing idea. Take $u > 0$ the first eigenfunction of $-\Delta + K$. Then $-\Delta u + Ku \geq u$. Consider a warped product $\tilde{g} = g + u^2 dt^2$[3]. Then we have that $\tilde{R} \geq 1$. Fix points $p, q \in \Sigma$. We minimize, among all unit-speed curves $\gamma : [0, l] \to \Sigma$ connecting p, q, the functional

$$\int_\gamma u ds = \int_0^l u(\gamma(s)) ds.$$

Equivalently, we minimize the area of $\gamma \times S^1 \subset (\Sigma \times S^1, \tilde{g})$. The stability inequality for the area functional implies that

$$\int_{\gamma \times S^1} \left[\frac{1}{2}(\tilde{R} + |\tilde{\mathrm{II}}|^2) - \tilde{K}\right] \varphi^2 u dt ds \leq \int_{\gamma \times S^1} |\tilde{\nabla}\varphi|^2 u dt ds,$$

for all S^1 invariant compactly supported functions φ. Plug in $\tilde{R} \geq 1$ and $\tilde{K} = -\frac{u''}{u}$ and get (throw away the $|\mathrm{II}|^2$ term):

$$\int_0^l \varphi^2 u ds + \int_0^l \frac{u''}{u} \varphi^2 u ds \leq \int_0^l (\varphi')^2 u ds.$$

As before, set $\varphi = u^{-\frac{1}{2}}\psi$, integrate by parts and use the AM-GM inequality to bound all terms involving u, we find:

$$\int_0^l \psi^2 \leq \frac{3}{4} \int_0^l (\psi')^2.$$

But we know $\lambda_1(-\frac{d^2}{dt^2}) = \frac{\pi^2}{l^2}$. This gives $l \leq \frac{2}{\sqrt{3}}\pi$.

□

This gives a more quantitative control of minimal slicings: given

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset (M^n, g)$$

a weighted minimal slicing with $R_g \geq 1$, we have:

(1) If $k = 2$, then Σ_2 is a disjoint union of S^2 with area $\leq 4\pi$ and diameter $\leq \frac{2}{\sqrt{3}}\pi$.
(2) If $k = 1$, then Σ_1 is a disjoint union of S^1 with length $\leq \frac{4}{\sqrt{3}}\pi$.

[3]For the more general version, one considers $\tilde{g} = g + u^{2a} dt^2$.

3.2. μ**-bubbles.** We have seen that a crucial property on a submanifold we seek is the eigenvalue condition $\lambda_1(-\Delta + \frac{1}{2}R) \geq \lambda > 0$. So far, we rely on stable minimal hypersurfaces to guarantee this condition. However, minimal surfaces (let along stable minimal surfaces) do not always exist under the PSC condition. To illustrate this, consider a simple situation where $M \simeq \Sigma\times[-1,1]$. One may not find any minimal surface at all without appropriate assumptions on the boundary $\Sigma \times \{\pm 1\}$.

In a seminal work, Gromov [Gro20] proved the following band width estimate for manifolds with scalar curvature.

Theorem 3.6 (Gromov [Gro20]). *Let $2 \leq n \leq 6$. Suppose g is a metric on $T^n \times [-1,1]$ satisfying $R_g \geq n(n+1)$. Then*

$$d_g(T^n \times \{-1\}, T^n \times \{1\}) \leq \frac{2\pi}{n+1}.$$

Remark 3.7. In Theorem 3.6, no boundary conditions are assumed along $T^n \times \{\pm 1\}$.

Remark 3.8. The constant $\frac{2\pi}{n+1}$ is sharp, as illustrated by the following example. Let g be a flat metric on T^n, $u \in C^\infty([-1,1])$. Recall from (1.4) that the metric $\tilde{g} = u^2 g + dt^2$ has scalar curvature

$$R_{\tilde{g}} = -2n\frac{u''}{u} - n(n-1)\frac{(u')^2}{u^2}.$$

Setting $h = n\frac{u'}{u}$ (note that $h(t)$ is the mean curvature of $T^n \times \{t\}$), we have that

$$R_{\tilde{g}} + 2h' + \frac{n+1}{n}h^2 = 0. \tag{3.1}$$

If we ask that $R_{\tilde{g}} = n(n+1)$, then a solution to (3.1) is given by

$$h(t) = -n\tan\left(\frac{n+1}{2}t\right) \quad \Rightarrow \quad u(t) = \left(\cos(\frac{n+1}{2}t)\right)^{\frac{2}{n+1}}.$$

Note that $u > 0$ on the interval $(-\frac{\pi}{n+1}, \frac{\pi}{n+1})$, having length $\frac{2\pi}{n+1}$.

Gromov's idea is to find a hypersurface that minimizes a prescribed mean curvature functional, trading minimality for existence in more general situations.

Given $M = T^n \times [-1,1]$, denote by $M_- = T^n \times \{-1\}$, $M_+ = T^n \times \{1\}$. For an open set Ω containing $\partial_- M$, let Σ be the hypersurface defined as $\partial\Omega = \Sigma - \partial_- M$ (here we are treating each term as oriented objects). Among all such Ω, we minimize

$$\mathcal{A}(\Omega) = |\Sigma| - \int_\Omega h,$$

for some $h \in C^\infty(M)$.

Theorem 3.9 (Existence of minimizer)**.** *Suppose $2 \leq n \leq 6$. Equip $\partial_\pm M$ with unit normal vector fields pointing the same was as ∂_t. Suppose that*

$$h|_{\partial_- M} > H_{\partial_- M}, \quad h|_{\partial_+ M} < H_{\partial_+ M}.$$

Then $\mathcal{A}(M)$ is minimized by a set $\Omega \subset M$ with smooth boundary, and Σ is disjoint from $\partial_\pm M$.

Sketch of proof. The key is to show that the minimizer Ω of $\mathcal{A}$ separates $\partial_- M$ and $\partial_+ M$. To do this, one checkes that by adding a neighborhood of $\partial_- M$ or removing a neighborhood of $\partial_+ M$, $\mathcal{A}$ is decreased. □

The separating hypersurface Σ is called a μ-bubble after Gromov. We compute the first and second variation of μ-bubbles. Deform Σ in the normal direction by vector fields $f_t \nu_t$. This also generates a variation Ω_t of Ω.

Theorem 3.10. *We have:*

$$\frac{d}{dt}\mathcal{A}(\Omega_t) = \int_{\Sigma_t} (H - h) f dA. \tag{3.2}$$

If Ω is stationary for $\mathcal{A}$, then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{A}(\Omega_t) = \int_\Sigma |\nabla f|^2 - (\mathrm{Ric}_M(\nu,\nu) + |\,\mathrm{II}\,|^2 + \langle \nabla_M h, \nu\rangle) f^2. \tag{3.3}$$

From (3.2), any $\mathcal{A}$-stationary Ω satisfies that $H_\Sigma = h|_\Sigma$. Thus, the $\mathcal{A}$ functional is usually called the prescribed mean curvature functional. The key point is to combine the second variation, (3.3) with the Schoen-Yau rearrangement trick (2.1) as follows. On Σ, we have that

$$|\,\mathrm{II}\,|^2 \geq \tfrac{1}{n}(\mathrm{tr}_g \mathrm{II})^2 = \tfrac{1}{n} H^2 = \tfrac{1}{n} h^2.$$

Thus, (3.3) implies that

$$\int_\Sigma |\nabla f|^2 \geq \int_\Sigma \left[\frac{1}{2}(R_M - R_\Sigma + \frac{n+1}{n} h^2) + \langle \nabla_M h, \nu\rangle\right] f^2,$$

$$\Rightarrow \int_\Sigma |\nabla f|^2 + \frac{1}{2} R_\Sigma f^2 \geq \frac{1}{2}\int_\Sigma \left(n(n+1) + \frac{n+1}{n} h^2 + 2\langle \nabla_M h, \nu\rangle\right) f^2. \tag{3.4}$$

Therefore, suppose that we may choose a prescribing function $h \in C^\infty(\mathring{M})$ such that:

1. $h(p) \to \infty$ as $p \to \partial_- M$, $h(p) \to -\infty$ on $p \to \partial_+ M$;
2. $n(n+1) + \frac{n+1}{n} h^2 - 2|\nabla_M h| > 0$ everywhere.

Then Theorem 3.9 guarantees the hypersurface Σ satisfying (3.4), and hence

$$\lambda_1(-\Delta + \frac{1}{2} R_\Sigma) > 0.$$

In particular we know that Σ is Yamabe positive. On the other hand, the projection map $M = T^n \times [-1,1] \to T^n$, restricted on Σ, has degree one, contradicting Corollary 2.12.

Comparing with (3.1), we construct such a function h. Assume, for the sake of contradiction, that $\operatorname{dist}(\partial_- M, \partial_+ M) > L > \frac{2\pi}{n+1}$. Choose ρ a smoothing of $\operatorname{dist}(\cdot, \partial_- M)$ such that

$$\partial_- M = \rho^{-1}(0), \quad \partial_+ M = \rho^{-1}(L), \quad |\operatorname{Lip} \rho| \leq 1.$$

Set

$$h(p) = -n \tan\left(\frac{\pi}{L}\rho(p) - \frac{\pi}{2}\right).$$

By a direct computation,

$$|\nabla h| < \frac{n(n+1)|\operatorname{Lip}\rho|}{2\cos^2(\frac{\pi}{L}\rho - \frac{\pi}{2})},$$

and thus $n(n+1) + \frac{n+1}{n}h^2 - 2|\nabla h| > 0$. This finishes the proof.

4. Applications and open questions

In this section we discuss some applications of these quantitative estimates.

4.1. Apherical manifolds.

Since the solution of the Geroch conjecture, there have been extensive investigations on various generalizations. A well-known conjecture in this direction is the following.

Conjecture 4.1. A closed aspherical manifold M^n does not admit any Riemannian metric with positive scalar curvature.

Recall that a manifold M^n is called aspherical, if for all $k \geq 2$, $\pi_k(M) = 0$. Equivalently, the universal cover $\tilde{M}$ is contractible. Since the only nontrivial homotopy group of M is the fundamental group, such M is the Eilenberg-MacLane space $K(\pi, 1)$ of its fundamental group π.

Example 4.2.
(1) *T^n is aspherical, since its universal cover, $\mathbf{R}^n$, is contractible.*
(2) *All hyperbolic manifolds (i.e. manifolds admitting a metric with constant sectional curvature -1) are aspherical, since they are all covered by $\mathbf{H}^n$.*
(3) *More generally, if a manifold M^n admits a metric g with nonpositive sectional curvature, then it is aspherical. Indeed, the universal cover is diffeomorphic to $\mathbf{R}^n$ by the Cartan-Hadamard theorem.*

Aspherical manifolds exist in abundance, and they are important objects in algebraic topology. Rosenberg [Ros83, Theorem 3.5] proved that a certain version of the (still open) strong Novikov conjecture implies Conjecture 4.1.

Let us try to get a feeling of the statement in low dimensions. When $n = 3$, we have the following decomposition theorem:

Theorem 4.3 (Kneser, Milnor ([Mil62]))**.** *Any closed* 3*-manifold* M *can be uniquely decomposed into prime factors:*

$$M = X_1 \# \cdots \# X_a \# \left(\#_1^b S^2 \times S^1 \right) \# K_1 \# \cdots \# K_c,$$

where each X_i *has finite fundamental group, and each* K_j *has universal cover diffeomorphic to* $\mathbf{R}^3$*.*

Remark 4.4. The resolution of Poincaré conjecture implies that each X_i is diffeomorphic to S^3/Γ_j. More generally, Thurston's geometrization gives a full classification of the K_j factors, but we won't need to use it here.

The relevance of the aspherical 3-manifolds and scalar curvature can be highlighted in the following result.

Theorem 4.5 (Schoen-Yau [SY79a], Gromov-Lawson [GL80])**.** *If a closed oriented* 3*-manifold admits a PSC metric, then there is no aspherical factors in its prime decomposition.*

In our first application, let us prove this Conjecture 4.1 for 3-manifolds.

Theorem 4.6. *An aspherical* 3*-manifold does not admit a metric with positive scalar curvature.*

We begin with the following topological facts. Let (M^n, g) be a closed aspherical manifold. The following facts hold.

(1) The universal cover $\tilde{M}$ is noncompact. This is because any connected compact n-manifold X satisfies that $H_n(X, \mathbf{Z}_2) = \mathbf{Z}_2$.
(2) $(\tilde{M}, g)$ has a length-minimizing geodesic $\sigma : \mathbf{R} \to \tilde{M}$. The existence of such a geodesic line holds for all noncompact universal covers of closed manifolds.
(3) For each $L > 0$, there exists an $(n-2)$-dimensional cycle Γ whose linking number with σ equals 1, and $\mathrm{dist}(\Gamma, \sigma) > L$. This can be constructed by taking suitable intersections of $\sigma(\mathbf{R})$ and $\sigma((-\infty, 0])$.

Proof of Theorem 4.6. Without loss of generality assume $R_g \geq 2$. Take the geodesic line σ and the linking 1-cycle γ in $(\tilde{M}, g)$ (with slight abuse of notation, we use g to denote the pull back metric in $\tilde{M}$ by the covering map) as above, with $L > \frac{2}{\sqrt{3}}\pi$. Since $\tilde{M}$ is simply connected, γ is null-homologous. We take Σ to be the area-minimizing surface with boundary Σ. Then $\Sigma \cap \sigma \neq \emptyset$. By the stability inequality, we have that

$$\lambda_1(-\Delta_\Sigma + K_\Sigma) \geq \frac{1}{2} R_{\tilde{M}} \geq 1.$$

By Theorem 3.4, for every $p \in \Sigma$,

$$\mathrm{dist}_\Sigma(p, \gamma = \partial\Sigma) \leq \frac{2}{\sqrt{3}}\pi.$$

However, this means that $\sigma \cap \Sigma = \emptyset$, contradiction. $\square$

Conjecture 4.1, when $n \in \{4, 5\}$, was proved by Chodosh-Li [CL24] and independently by Gromov [Gro20]. We will sketch a proof here for the case $n = 4$.

Proof of Conjecture 4.1 when $n = 4$. Without loss of generality assume that $R_g \geq 4$. Take the same construction of a linking geodesic line σ and a 2-cycle Σ_0 in the universal cover $(\tilde{M}, g)$, such that $d_{\tilde{M}}(\sigma, \Sigma_0) > L$ for some L to be chosen later.

As before, we take a homologically minimizing hypersurface N^3 such that $\partial N = \Sigma_0$. The stability inequality implies that $\lambda_1(-\Delta_N + \frac{1}{2}R_N) \geq 2$. Now we run into an issue: a three-manifold with $R_N \geq 4$ may have arbitrarily large diameter, so a direct analogy of the previous proof does not work.

The idea is to use μ-bubbles. For simplicity we first prove the following.

Proposition 4.7. *Suppose (N^3, g) has $R_g \geq 4$, $\partial N = \Sigma_0$ is commpact. Then there exists $\Sigma^2 \subset N^3$ homologous to Σ_0 in N such that:*

(1) $\operatorname{dist}_N(\Sigma, \Sigma_0) \leq 4\pi$, *and*
(2) *each connected component of Σ has diameter $\leq \frac{2}{\sqrt{3}}\pi$.*

The proof of Proposition 4.7 is an application of μ-bubbles.

Proof of Proposition 4.7. Consider a smooth domain $N_0 \subset N$ containing $\partial N = \Sigma_0$, such that writing $\partial N_0 = \Sigma_0 \sqcup \Sigma_1$, we have that $4\pi - \varepsilon \leq \operatorname{dist}_N(\Sigma_0, \Sigma_1) \leq 4\pi$ (the extra room with ε is to guarantee that ∂N_0 is smooth).

Let's construct a function $h \in C^\infty(\mathring{N}_0)$ such that:

(1) $h(p) \to +\infty$ as $p \to \Sigma_0$, $h(p) \to -\infty$ as $p \to \Sigma_1$;
(2) $1 + \frac{3}{2}h^2 - 2|\nabla h| \geq 0$.

The construction of such h is as follows. Take ρ a smoothing of $\operatorname{dist}_N(\Sigma_0, \cdot)$ with $|\operatorname{Lip}\rho| < 2$, such that $\Sigma_0 = \rho^{-1}(0)$, $\Sigma_1 = \rho^{-1}(2\pi)$. This can be arranged by the distance assumption between Σ_0 and Σ_1. Then define

$$h = -\tan\left(\frac{\rho - \pi}{2}\right).$$

It is simple to check that both conditions (1) and (2) above are satisfied. Therefore, the functional

$$\mathcal{A}(\Omega) := |\partial\Omega| - \int_\Omega h$$

has a smooth minimizer Ω, such that $\Omega = \Sigma - \Sigma_0$ with

$$\int_\Sigma |\nabla f|^2 + \frac{1}{2}R_\Sigma f^2 \geq \int_\Sigma \frac{1}{2}R_N + \frac{3}{2}h^2 + 2\langle \nabla h, \nu\rangle f^2 \geq 0$$

for all $f \in C^\infty(\Sigma)$. Using that $R_N \geq 4$ and (2), we conclude that

$$\lambda_1(-\Delta + K_\Sigma) \geq 1,$$

and hence Theorem 3.4 concludes that each connected component has diameter $\leq \frac{2}{\sqrt{3}}\pi$. ☐

To finish the proof, need to utilize another quantitative topological property of the universal cover $(\tilde{M}, g)$.

Lemma 4.8 (Uniform filling)**.** *Let (M, g) be a closed Riemannian manifold, and $H_k(\tilde{M}) = 0$. Then there exists an increasing function $\Lambda : \mathbf{R}_+ \to \mathbf{R}_+$ such that any k-cycle X in $\tilde{M}$ with $\mathrm{diam}_{\tilde{M}}(X) \leq r$ can be written as $X = \partial Y$ for some $(k+1)$-chain with $\mathrm{diam}_{\tilde{M}}(Y) \leq \Lambda(r)$.*

Proof of Lemma 4.8. Fix a point $p \in \tilde{M}$. Translate X with some deck transformation so $d_{\tilde{M}}(p, X) \leq D$, where $D = \mathrm{diam}(M)$. Then $X \subset B_{\tilde{M}}(p, D + r)$. Note that $H_k(B_{\tilde{M}}(p, D + r))$ is finitely generated. Pick a set of generators $[X_1], \cdots, [X_N]$. Write each $X_j = \partial Y_j$ and set $\Lambda(r) = 2 \max_j \mathrm{dist}_{\tilde{M}}(p, Y_j)$. Then we may write X as the boundary of a $(k+1)$-chain of diameter $\leq 2\Lambda(r)$. ☐

Now let's choose $L > 4\pi + \Lambda(\frac{2}{\sqrt{3}}\pi)$. Note that $\partial\Omega = \Sigma - \Sigma_0$. Since each connected component of Σ has diameter $\leq \frac{2}{\sqrt{3}}\pi$, we use Lemma 4.8 can write $\Sigma = \partial\Omega_1$ for $\Omega \subset N_{\Lambda(\frac{2}{\sqrt{3}}\pi)}$. Then $\Sigma_0 = \partial(\Omega + \Omega_1)$, and we have that

$$\mathrm{dist}_{\tilde{M}}(\Omega + \Omega_1, \Sigma_0) < 4\pi + \Lambda\left(\frac{2}{\sqrt{3}}\pi\right) < L.$$

This implies that Σ_0 and σ does not link, contradiction.

☐

4.2. **Urysohn width and macroscopic dimension.**

Gromov (see, e.g. [Gro86, Gro17, Gro19]) proposed to study PSC manifolds via notions of macroscopic geometry. These notions measure the size of a Riemannian manifold. For example:

Definition 4.9. Let (M, d) be a metric space, $k \in \mathbf{Z}_+$. We say that the k-th Urysohn width of (M, d) is bounded by $\Lambda < \infty$, if there exists a k-dimensional simplicial complex P and a continuous map $f : (M, d) \to P$, such that

$$\mathrm{diam}(f^{-1}(p)) \leq \Lambda, \quad \forall p \in P.$$

Intuitively, a Riemannian manifold (M^n, g) has Urysohn k-width bounded means that (M^n, g) is close to a k-dimensional space. We easily deduce from the definition that if a metric space has Urysohn k-width $\leq \Lambda$, then for every $k' > k$, its Urysohn k'-width is $\leq \Lambda$.

Definition 4.10. Let (M, d) be a metric space. Its macroscopic dimension, denoted by $\dim_{mc}(M)$, is the smallest integer k such that the Uryshon k-width of M is finite.

Gromov has made the following deep conjecture relating the Urysohn width and positive scalar curvature.

Conjecture 4.11 (Gromov [Gro17])**.** For each $n \geq 2$, there exists a constant $c(n)$ such that any closed Riemannian manifold (M^n, g) with $R_g \geq 1$ has Urysohn $(n-2)$-width $\leq c(n)$.

Recently, this conjecture was answered affirmatively by Liokumovich-Maximo [LM23]. Slicing with μ-bubbles, we can also prove this for simply connected PSC 3-manifolds.

4.3. **Extension to other curvature conditions.** Recently, Brendle-Hirsch-Johne [BHJ24] defined and investigated a series of curvature notions interpolating between Ricci and scalar curvature. For a Riemannian manifold (M^n, g) and an integer $m \leq n-1$, the m-intermediate curvature is defined for an unordered pair of m orthonormal tangent vectors at a point p $e_1, \cdots, e_m \in T_pM$:

$$C_m(e_1, \cdots, e_m) = \sum_{p=1}^{m} \sum_{q=p+1}^{n} R^M(e_p, e_q, e_q, e_p),$$

where $\{e_j\}_{j=1}^n$ is an extension of $e_1, \cdots, e_m$ to an orthonormal basis. We note that C_1 is the Ricci curvature, C_2 is called the BiRicci curvature, and C_{n-1} is a equivalent to scalar curvature. In [BHJ24], an interesting dimension descent property was discovered for the C_m curvature.

Theorem 4.12 (Brendle-Hirsch-Johne [BHJ24])**.** *Assume* $1 \leq m \leq n-1$, $n(m-2) \leq m^2 - 2$. *Suppose* (N^n, g) *is closed and the* m*-intermediate curvature of* g *is positive. Then* N *admits no weighted minimal slicing*

$$\Sigma_{n-m} \subset \cdots \subset \Sigma_{n-1} \subset N^n.$$

In particular, this shows that for any X, $X^{n-m} \times T^m$ admits no metric with positive m-intermediate curvature.

One may speculate more quantitative versions of Theorem 4.12. In particular, there have been interesting developments in understanding the BiRicci curvature.

Theorem 4.13 ([CLMS24],[AX24])**.** *Let* $n \leq 5$. *There exists a constant* $c(n)$ *such that for all* (M^n, g) *a complete, simply connected Riemannian manifold with BiRicci curvature* ≥ 1*, its Urysohn* 1*-width bounded by* $c(n)$*.*

Such bounds have surprising applications to other geometric problems, including the recent advance in the stable Bernstein problem for minimal hypersurfaces (see, e.g. [CLMS24]).

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Courant Institute, New York University, 251 Mercer St, New York, NY 10012, USA

Email address: `chaoli@nyu.edu`