NON-EXISTENCE OF METRIC ON T^n WITH POSITIVE SCALAR CURVATURE

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Abstract. In this note we present Gromov-Lawson's result on the non-existence of metric on T^n with positive scalar curvature.

Historical results

Scalar curvature is one of the simplest invariants of a Riemannian manifold. In general dimensions, this function (the average of all sectional curvatures at a point) is a weak measure of the local geometry, hence it's suspicious that it has no relation to the global topology of the manifold. In fact, a result of Kazdan-Warner in 1975 states that on a compact manifold of dimension ≥ 3, every smooth function which is negative somewhere, is the scalar curvature of some Riemannian metric. However people know that there are manifolds which carry no metric whose scalar scalar curvature is everywhere positive.

The first examples of such manifolds were given in 1962 by Lichnerowicz. It is known that if X is a compact spin manifold and $\hat{A} \neq 0$ then by Lichnerowicz formula (which will be discussed later) then X doesn't carry any metric with everywhere positive scalar curvature. Note that spin assumption is essential here, since the complex projective plane has positive sectional curvature and non-zero $\hat{A}$-genus.

Despite these impressive results, one simple question remained open: Can the torus T^n, $n \geq 3$, carry a metric of positive scalar curvature? This question was settled for $n \leq 7$ by R. Schoen and S. T. Yau. Their method involves the existence of smooth solution of Plateau problem, so the dimension restriction $n \leq 7$ is essential. So the method cannot be generalized easily to general dimension.

This result by Gromov and Lawson is deduced by a brilliant application of the Atiyah-Singer index theorem for a twisted spinor bundle on a spin manifold. It successful settled the general question about T^n, and also proved the non-existence of metric with positive scalar curvature on a large class of manifolds that are called solvmanifolds.

1. Spin manifolds and Lichnerowicz formula

Definition 1.1. Let M^n be a oriented Riemannian manifold, $\pi_1 : F(M) \to M$ is the principel $SO(n)$-bundle arised by the Riemannian metric. A spin structure of M is a principle Spin(n)-bundle $\pi_2 : SP(M) \to M$ together with a bundle map $f : SP(M) \to F(M)$ such that

(1) For any $p \in SP(M)$, $\pi_1(f(p)) = \pi_2(p)$.

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(2) For any $p \in SP(M)$, $g \in \mathrm{Spin}(n)$, we have $f(p \cdot g) = f(p) \cdot \sigma(g)$, where $\sigma : \mathrm{Spin}(n) \to SO(n)$ is the double cover.

If M has a spin structure we call M a spin manifold.

From the exact sequence of groups

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 1,$$

we get the long exact sequence of Cech cohomology of M:

$$\ldots \to H^1(M, \mathbb{Z}_2) \to H^1(M, \mathrm{Spin}(n)) \to H^1(M, SO(n)) \to H^2(M, \mathbb{Z}_2).$$

So M has a spin structure if and only if the image of map $w_2 : H^1(M, SO(n)) \to H^2(M, \mathbb{Z}_2)$ vanishes. This is equivalent to the second Stiefel-Whitney class $w_2(M)$ is equal to 0. In this case the different spin structures of M is classified by elements in $H^1(M, \mathbb{Z}_2)$.

Suppose M is a spin manifold with a fixed spin structure. Let S be the space of spinors, with $\rho : \mathrm{Spin}(n) \to \mathrm{Aut}(S)$ being the unitary complex representation of $\mathrm{Spin}(n)$. Then we construct a vector bundle $S(M) = SP(M) \times_\rho S$, called the spinor bundle of M. Note that since S is a left module of the real Clifford algebra C_n, $S(M)$ is a Clifford bundle in the sence that for any $\varphi \in Cl(M)$, $\sigma \in S(M)$, $\varphi \cdot \sigma$ is an element in $S(M)$.

The unique Levi-Civita connection on $T(M)$ defines a unique connection 1-form on $F(M)$, hence it's pulled back by f to be an orthogonal connection ω on $SP(M)$. ω decends again to be an orthogonal connection on the associated bundle $S(M)$. We also denote this connection by ∇. If furthermore $E \to M$ is an Hermitian bundle with connection ∇^E, then on the twisted spinor bundle $S \otimes E$ we have a connection defined by

$$\nabla(\sigma \otimes e) = \nabla\sigma \otimes e + \sigma \otimes \nabla^E e.$$

Definition 1.2. On a twisted spinor bundle $S \otimes E$, suppose $\{e_i\}$ is an orthonormal frame. Define Dirac operator to be

$$D = \sum_i e_i \cdot \nabla_{e_i}.$$

Here $\cdot$ is the Clifford multiplication.

Remark 1.3. Dirac opeartor is a first order elliptic operator, and it's essentially self-adjoint, meaning that for any two sections of the twisted spinor bundle s_1, s_2, $\int \langle Ds_1, s_2\rangle = \int \langle s_1, Ds_2\rangle$. And it can be uniquely extended to be a self-adjoint operator from $L^2(S(M) \otimes E) \to L^2(S(M) \otimes E)$.

The importance of Dirac lies in the following formula.

Theorem 1.4. *Suppose $S \otimes E$ is a twisted spinor bundle over a spin manifold M. Then we have*

$$D^2 = -\sum_i \nabla^2_{e_i, e_i} + \sum_{i<j} e_i \cdot e_j \cdot \mathcal{R}_{ij} \tag{1.1}$$

here $\mathcal{R}_{ij}$ is the curvature operator of the bundle $S(M) \otimes E$.

Proof. We omit the proof. It's rather direct computation. □

Lemma 1.5. *Suppose M is spin, φ is a smooth section of $S(M)$. Then $\mathcal{R}_{i,j}\varphi = \frac{1}{4}\sum_{k,l} R_{ijkl} e_k \cdot e_l \cdot \varphi$.*

Proof. This is a result given by the relation between the curvature operator on an associate bundle and the curvature form on the principal bundle. In general, assume $P \to M$ is a principal G-bundle and $f : G \to \mathrm{Aut}(B)$ is a representation of G, and $B^0 = P \times_f B$ is the associated bundle. Let $f' : \mathfrak{g} \to \mathrm{End}(B)$. Then for any $(x, b) \in B^0$, V, W tangent vectors on M, we have

$$\mathcal{R}^0_{VW} b = f'(\Omega(\tilde{V}, \tilde{W}))b.$$

In our case the curvature form on $F(M)$ is given by $\tilde{\Omega} = \sum_{i<j} \Omega^i_j E^j_i$, where E^j_i is the map defined by $e_i \mapsto e_j$, $e_j \mapsto -e_i$ and $e_k \mapsto 0$ for $k \neq i, j$. On $SP(M)$ the curvature is the pull back of $\tilde{\Omega}$ by f', hence is $\Omega = \frac{1}{2} \sum_{i<j} \Omega^i_j e_i \cdot e_j$. Then for two tangent vectors V, W and a section S, we have

$$\mathcal{R}_{V,W} S = \frac{1}{2} \sum_{k<l} \Omega^k_l (E_i, E_j) V_k \cdot V_l \cdot S.$$

However we know in Riemannian geometry that $\Omega^k_l(E_i, E_j) = R_{ijkl}$, the lemma got proved. $\square$

Proposition 1.6. *Let M be spin manifold and $S(M)$ be the spinor bundle. Then*

$$\mathcal{R} = \frac{1}{4}\kappa.$$

κ is the scalar curvature of M.

Proof.

$$\begin{aligned}
\mathcal{R} &= \frac{1}{2} \sum_{i,j} e_i \cdot e_j \mathcal{R}_{e_i, e_j} \\
&= \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} e_i e_j e_k e_l \\
&= \frac{1}{8} \sum_l \left(\frac{1}{3} \sum_{i,j,k \text{distinct}} (R_{ijkl} + R_{jkil} + R_{kijl}) e_i e_j e_k + \sum_{i,j} R_{ijil} e_i e_j e_i + \sum_{i,j} R_{ijjl} e_i e_j e_j \right) e_l \\
&= \frac{1}{4} \sum_{i,j,l} R_{ijil} e_j e_l \\
&= \frac{1}{4}\kappa
\end{aligned}$$

$\square$

Combining the results above, we have

Theorem 1.7 (Lichnerowicz)**.** *Let M be a spin manifold, $S(M) \otimes E$ be a twisted spinor bundle. Then*

$$D^2 = -\sum_i \nabla_{e_i, e_i} + \frac{1}{4}\kappa + \mathcal{R}^E.$$

Proof. The curvature operator of $S(M) \otimes E$ is given by

$$\mathcal{R}(\sigma \otimes e) = (\mathcal{R}^{S(M)}) \otimes e + \sigma \otimes (\mathcal{R}^E e).$$

Hence by what we know now,

$$\mathcal{R}(\sigma \otimes e) = \frac{1}{4}\kappa\sigma \otimes e + \frac{1}{2}\sum_{j,k}(e_j \cdot e_k \cdot \sigma) \otimes (\mathcal{R}^E_{e_j,e_k} e).$$

□

2. Vanishing theorems

Let M be a compact spin manifold of dimension $2n$. On the space of spinors S, the element $\omega = i^n e_1 \dots e_{2n}$ is well defined and satisfies $\omega^2 = 1$ and $e_j\omega = -\omega e_j$ for all j. Therefore there is a decomposition $S = S^+ \oplus S^-$ of $+1$ and -1 eigenspaces of multiplication by ω. Clearly Clifford multiplication by e_j sends S^+ to S^- and S^- to S^+. Reflected on the twisted spinor bunlde, this gives

$$S(M) \otimes E = S^+ \oplus S^-,$$

and since further $\nabla\omega = 0$, the Dirac operator maps $\Gamma(S^+)$ to $\Gamma(S^-)$ and $\Gamma(S^-)$ to $\Gamma(S^+)$. Therefore the restriction of D onto $\Gamma(S^+)$, we call which D^+, is an elliptic operator $\Gamma(S^+) \to \Gamma(S^-)$ with adjoint operator $D^- = D|_{S^-}$.

The celebrated Atiyah-Singer theorem states that

Theorem 2.1 (Atiyah-Singer)**.** *Let M be a spin manifold of dimension $2n$, with a twisted spinor bundle $S(M) \otimes E$. Then*

$$\mathit{Index}(D^+) = \{\mathit{ch}E \cdot \hat{\mathbf{A}}(M)\}[M].$$

Here $\hat{\mathbf{A}}(M)$ denoted the total $\hat{\mathbf{A}}$ genus of M.

By Atiyah-Singer and theorem 1.7, we have the following vanish theorem

Theorem 2.2. *Let M be a compact spin manifold of even dimension with twisted spinor bundle $S(M) \otimes E$. If $\kappa > 4\mathcal{R}^E$, then $\ker(D^+)$ and coker(D^+) are zero. In particular, if $\kappa > 4\mathcal{R}^E$, then $\{\mathit{ch}E \cdot \hat{\mathbf{A}}(M)\}[M] = 0$.*

Corollary 2.3. *Let M be a spin manifold with nonzero $\hat{A}$-genus, then M cannot carry a metric with quasi-positive scalar curvature.*

3. Proof of Main theorems

Definition 3.1. A C^1-map $f : M \to N$ between Riemannian manifolds is called ϵ-contracting if for any tangent vectors v to M we have $\|f_* v\| \le \epsilon\|v\|$.

Since curvature conditions are local properties, it's natural to consider the Riemannian covering of a manifold with given curvature conditions. This invokes the following definition.

Definition 3.2. A complete connected oriented compact Riemannian n-manifold M is said to be enlargeable if for any $\epsilon > 0$, there exitst an orientable Riemannian covering space which admits an ϵ-contracting map onto S^n which is constant at infinity and of non-zero degree.

If for each $\epsilon > 0$ there is a finite convering space with these properties, we call the manifold compactly enlargeable.

The next theorem shows that enlargeability is an almost topological property of a manifold.

Theorem 3.3. *The following statement hold in the category of compact manifolds:*

(1) *Enlargeability is independent of the Riemannian metric.*
(2) *Enlargeability depends only on the homotopy type of the manifold.*
(3) *The product of enlargeable manifolds is enlargeable.*
(4) *The connected sum of any manifolds with an enlargeable manifold is again enlargeable.*
(5) *Any manifold which admits a map of non-zero degree onto an enlargeable manifold is itself enlargeable.*

Proof. Omitted. □

The following theorem illustrates a large class of enlargeable manifolds.

Proposition 3.4. *Suppose M is a compact Riemannian n-manifold with a non-positive sectional curvature. Then M is enlargeable.*

Furthermore, if $\pi_1(M)$ satisfies the condition that, for any finitely many elements $g_1, \dots, g_N$ in $\pi_1(M)$, there exists a normal subgroup N of $\pi_1(M)$ of finite index such that $g_i \notin N$ for all i, then M is compactly enlargeable.

Proof. M is so-called Cartan-Hadamard manifold, so the universal covering $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$ via $e^{-1} : \tilde{M} \to T_p(M)$ where e is the exponential map at $p \in T_p(\tilde{M})$. Since sectional curvature of M is non-positive, e^{-1} is 1-contracting. Choose a degree-1 map $\phi : \mathbb{R}^n \to S^n$ which is constant outside the Euclidean ball B_1 of radius 1. Clearly ϕ is α-contracting for some positive real α. Set $\phi_r : \tilde{M} \to S^n$ to be $\phi_r(x) = \phi(r \cdot e^{-1}(x))$. Then ϕ_r is αr-contracting and constant outside $e(B_{1/r})$. This proves the first part of the proposition.

For the second part, let $F \subset \tilde{M}$ be a fundamental domain for the deck transformation of $\pi_1(M)$ on M. Then for each fixed r there is a finte set of group elements $g_1, \dots, g_N \in \pi_1(M)$ such that $e(B_{1/r}) \subset \cup_{i=1}^N g_i(F)$. By assumption there exitsts a normal subgroup π' of finite index, and $g_i \notin pi'$. Let $M' \to M$ be the finte covering corresponding to π'. Then there is a fundamental domain $F' \subset \tilde{M}$ and $e(B_{1/r}) \subset F'$. Then ϕ_r descends to an αr-contracting map $\phi'_r : M' \to S^n$ of degree 1. □

Remark 3.5. T^n is compactly enlargeable because $\pi_1(T^n) = \mathbb{Z}^n$ satisfies the condition of the proposition. In fact, any "solvmanifold", i.e., a compact manifold diffeomorphic to G/Γ, where G is a solvable Lie group and Γ is a discrete subgroup, is enlargeable.

Theorem 3.6. *A compactly enlargeable spin manifold M cannot carry a metric of positive scalar curvature.*

Before proving the theorem, let me point out that the condition of being compactly enlargeable is not necessary. The same conclusion is true for any enlargeable manifold. But the proof requires relative index theorem, which will be more technically complicated.

Proof. Suppose M carries a metric with $\kappa \geq \kappa_0$ for some constant $\kappa_0 > 0$. WLOG we can assume M has dimension $2n$. (If M is of odd dimension, then consider $M \times S^1$ instead.) Choose a complex vector bundle E_0 over S^{2n} such that the $c_n(E_0)$, the top Chern class of E_0, does not vanish. This is always possible because on S^{2n} the Chern character

$$\mathrm{ch}E = \lambda_1 + \lambda_2 c_n(E)$$

gives an isomorphism $\mathrm{ch}: K(S^{2n}) \to H^*(S^{2n}, \mathbb{Z})$, here λ_1, λ_2 are constants. We fix an Hermitian connection ∇^{E_0} over E_0, and R^{E_0} denote the curvature 2-form.

Let $\epsilon > 0$ be given and choose a finite orientable covering $\tilde{M} \to M$ which admits an ϵ-contracting map to S^{2n} and of non-zero degree. Let $E = f^*E_0$ to be the pull back bundle of E_0 over $\tilde{M}$ with Hermitian connection. On the twisted spinor bundle $S(M) \otimes E$, by theorem 1.7,

$$D^2 = -\sum_i \nabla_{e_i,e_i} + \frac{1}{4}\kappa + \mathcal{R}^E.$$

Set $\|\mathcal{R}^E\| = \sup\{\langle \mathcal{R}^E(\varphi), \varphi\rangle : \|\varphi = 1\|\}$. Then there is a constant α depending only on dimension such that $\|\mathcal{R}^E\| \le \alpha \|R^E\|$. Further, the connection of E is the induced connection from E_0, so we have $R^E_{v,w} = R^{E_0}_{f_*v, f_*w}$. Since the map f is ϵ-contracting, we conclude $\|R^E\| \le \epsilon^2 \|R^{E_0}\|$. Therefore $\|\mathcal{R}^E\| \le \alpha\epsilon^2$, where α depends only on the data of S^{2n}.

Now everywhere on $\tilde{M}$ we have $\kappa \ge \kappa_0$. Choose $\epsilon < \sqrt{\kappa_0/\alpha}$. Then by Lichnerowicz formula the kernel of the Dirac operator D is zero. By Atiyah-Singer index theorem, the topological index of the twisted bundle vanishes, or

$$\{\mathrm{ch}E \cdot \hat{\mathbf{A}}(\tilde{M})\}[\tilde{M}] = 0.$$

Now the Chern charater is related to the analytic function e^z of the curvature matrix, so it's easy to see that $\mathrm{ch}E = \dim E + \frac{1}{(n-1)!}c_n(E)$. By the property of Chern class, $c_n(E) = c_n(f^*E_0) = f^*(c_n(E_0)) \ne 0$. And by corollary 2.3 we have now that the $\hat{A}$-genus of $\tilde{M}$ vanishes. Therefore

$$\begin{aligned}
0 = \mathrm{Index}(D^+) &= \{\mathrm{ch}E \cdot \hat{\mathbf{A}}(\tilde{M})[\tilde{M}] \\
&= \{(\dim E \cdot \hat{A}(\tilde{M}) + \frac{1}{(n-1)!}c_n(E)\}[\tilde{M}] \\
&= \frac{1}{(n-1)!}c_n(E)[\tilde{M}] \\
&= \frac{1}{(n-1)!}f^*(c_n(E_0))[\tilde{M}] \\
&= \frac{1}{(n-1)!}\deg f \cdot c_n(E_0)[S^{2n}] \\
&\ne 0
\end{aligned}$$

Contradiction. $\square$

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