Lecture 11

We now consider an important example of a Fredholm operator that is not of the form $1+K$ with K compact. Thus the Fredholm operators are a genuinely larger class of operators than those of the form $1+K$, K compact.

Examples of this type are intimately related to the theory of so-called Riemann-Hilbert problems.

Let $S^1 = \{z : |z| = 1\}$ denote the unit circle in $\mathbb{C}$.

For any $n \geq 1$, let $\mathcal{H}^{(n)} = L^2(S^1, \frac{d\theta}{2\pi}; \mathbb{C}^n)$

$$= \left\{ f = (f_1, \dots, f_n)^T : \int_{-\pi}^{\pi} \| f(e^{i\theta}) \|^2_{\mathbb{C}^n} \frac{d\theta}{2\pi} < \infty \right\},$$

and let

$$\ell_2^{(n)} = \left\{ a = \{a_k\}_{k\in\mathbb{Z}} : a_k \in \mathbb{C}^n \text{ and } \sum_{-\infty}^{\infty} \|a_k\|^2_{\mathbb{C}^n} < \infty \right\}.$$

The Fourier transform $\mathcal{F}$ maps $\mathcal{H}^{(n)}$ unitarily onto $\ell_2^{(n)}$

$$\mathcal{H}^{(n)} \ni f \mapsto \mathcal{F}f = \hat{f} = \left\{ f_k : \int_{-\pi}^{\pi} e^{-ik\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} \right\} \in \ell_2^{(n)}$$

$$\| f \|^2_{\mathcal{H}^{(n)}} = \sum_{k=-\infty}^{\infty} \| f_k \|^2_{\mathbb{C}^n}$$

The inverse $\mathcal{F}^{-1}$ of $\mathcal{F}$ is given by the L^2 convergent series

(136.1) $$f = \mathcal{F}^{-1}\hat{f} = \sum_{-\infty}^{\infty} f_k e^{ik\theta} \equiv \lim_{K\to\infty} \sum_{-K}^{K} f_k e^{i\theta}$$

For $f \in H^{(n)}$, define the bounded complementary orthogonal projections in $H^{(n)}$

(136.2) $$P_+ f = \sum_{k=0}^{\infty} f_k e^{ik\theta}, \quad P_- f = \sum_{-\infty}^{-1} f_k e^{ik\theta}$$

(136.3) $$\|P_\pm\| = 1, \quad P_+ + P_- = 1_{H^{(n)}}, \quad P_+ P_- = 0, \quad P_\pm^* = P_\pm$$

Let

(136.4) $$H_\pm = \operatorname{ran} P_\pm$$

Clearly functions in $H_\pm$ have analytic continuations to $\{|z|<1\}$ and $\{|z|>1\}$ respectively and if $g \in P_- f$, then $g(z) = O(1/|z|)$ as $|z| \to \infty$.

Now let h be a continuous function from S^1 to $Gl(n, \mathbb{C})$, the invertible $n \times n$ matrices with complex entries. Define the <u>Toeplitz operator</u> $T: H_+ \to H_+$:

(137.0) $$Tf \equiv P_+(hf) \qquad f \in H_+ .$$

(137.1) Clearly $T \in \mathcal{L}(H_+)$ and $\|T\| \leq n\|h\|_\infty$

We will prove the following result. Let $w(h)$ be the winding number of $\det h(z)$ on S^1, i.e.,

$$w(h) = \frac{1}{2\pi i}\,\Delta \log \det h$$

$$= \frac{1}{2\pi i}\left[\log \det h(e^{2\pi i}) - \log \det h(e^{i0})\right]$$

Proposition 137.2

The operator T in (137.1) is Fredholm and

(137.3) $$\operatorname{ind} T = -w(h)$$

Remark: If $w(h) \neq 0$, then clearly T cannot be of the form $1 + \text{compact}$.

Definition 137.4 Two nowhere singular continuous $n \times n$ matrix valued functions $h_1(z)$ and $h_2(z)$ on $S^1 = \{|z| = 1\}$ are homotopy equivalent if there exists a continuous map $h(z,t)$ from $S^1 \times [0,1]$ to $GL(n, \mathbb{C})$ with the property that

$h(z,0) = h_1(z)$ and $h(z,1) = h_2(z)$. We say that $h(z,t)$ provides a homotopy deformation of $h_1(z)$ to $h_2(z)$.

<u>Lemma 138.1</u>

Two nowhere singular, continuous $n \times n$ matrix valued functions on S^1 are homotopy equivalent if and only if $w(h_1) = w(h_2)$.

<u>Proof</u>: The winding # is clearly conserved under a homotopy deformation, so it remains to show that if $w(h_1) = w(h_2)$, then h_1 and h_2 are homotopy equivalent. It suffices to show that if $h(z)$ is a non-singular continuous $n \times n$ matrix valued function on S^1 with $w(h) = N$, then h is homotopy equivalent to $\text{diag}(e^{iN\theta}, 1, \dots, 1)$.

<u>Step 1</u> By the Stone-Weierstrass Theorem, there exists an $n \times n$ matrix valued function $h^{(1)}(z)$ on S^1 such that

(139.1) $h^{(1)}(z)$ is a trigonometric polynomial

$$h^{(1)}(z) = \sum_{l=-N^{(1)}}^{N^{(1)}} a_l z^l$$

for some $N^{(1)} < \infty$ and suitable $n \times n$ matrices a_l

(139.2) $h_{t_1}(z) = h(z) + t_1 \left(h^{(1)}(z) - h(z) \right)$ is continuous and nowhere singular for all $0 \le t_1 \le 1$, and hence provides a homotopy from h to $h^{(1)}$. We have $w(h^{(1)}) = w(h)$.

<u>Step 2</u> For any $n \times n$ matrix $M = (M_{ij})$ let $\Delta_1(M), \Delta_2(M), \ldots, \Delta_n(M)$ denote the leading sub-determinants, $\Delta_1(M) = M_{11}$, $\Delta_2(M) = \det \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \ldots,$ $\Delta_n(M) = \det M$.

Now

$$p(t_2) = \prod_{j=1}^{n} \Delta_j \left(h^{(1)}(z=1) + t_2 \right)$$

is a monic polynomial in t_2 of degree $\frac{n(n+1)}{2}$. Clearly there exists $t_2^0 \ge 0$, arbitrarily small so that

(139.3) $p(t_2^0) \neq 0$, and

(139.4) $h^{(1)}(z) + t_2$ is nowhere singular on S^1 for all $0 \le t_2 \le t_2^0$.

We have $w(h^{(1)} + t_2^{\circ}) = w(h^{(1)}) = w(h)$.

<u>Step 3</u> The rational function

$$d(z) \equiv \prod_{j=1}^{n} \Delta_j \left(h^{(1)}(z) + t_2^{\circ}\right)$$

is non-trivial as $d(z=1) = p(t_2^{\circ}) \neq 0$, and hence $d(z)$ has at most a finite # of zeros. It follows that there exists $t_3^{\circ} \geq 0$, arbitrarily small, so that

(140.1) $d(z - t_3^{\circ})$ has no zeros on S^1, and

(140.2) $h^{(1)}(z - t_3) + t_2^{\circ}$ is non-singular on S^1 for all $0 \leq t_3 \leq t_3^{\circ}$.

We have

$$w(h^{(2)}) = w(h)$$

where

$$h^{(2)}(z) = h^{(1)}(z - t_3^{\circ}) + t_2^{\circ}.$$

<u>Step 4</u> As the leading sub-determinants $\Delta_1(z), \ldots, \Delta_n(z)$ of $h^{(2)}(z)$ are all non-zero on $S^{(1)}$, it follows by Gaussian elimination that we may express

(140.3) $$h^{(2)}(z) = L(z)\, D(z)\, U(z)$$

where $L(z)$, $D(z)$ and $U(z)$ are continuous on S^1 and

(141.1) $L(z) = I + \hat{L}(z)$, where $\hat{L}(z)$ is strictly lower triangular

(141.2) $U(z) = I + \hat{U}(z)$, where $\hat{U}(z)$ is strictly upper triangular, and

(141.3) $D(z) = \text{diag}\,(D_1(z), D_2(z), \ldots, D_n(z))$, where

$$\Delta_1(z) = D_1(z)$$
$$\Delta_2(z) = D_1(z)\, D_2(z)$$
$$\vdots$$
$$\Delta_n(z) = D_1(z) \cdots D_n(z) .$$

Set

$$h^{(2)}(z,t) = \left(I + (1-t)\hat{L}(z)\right) D(z) \left(I + (1-t)\hat{U}(z)\right),$$
$$0 \le t \le 1$$

Then clearly $h^{(2)}(z,t)$ gives a homotopy deformation from

$$h^{(2)}(z, t=0) = h^{(2)}(z) \quad \text{to} \quad h^{(2)}(z, t=1) = D(z)$$

In particular $w(D) = w(h)$

This fact is of course true, a fortiori, as $\det(D(z)) = \det h^{(2)}(z)$.

<u>Step 5</u> For $0 \le t \le 1$, consider

$$h^{(3)}(z,t) = \begin{pmatrix} D_1(z) & & & & \\ & \ddots & & 0 & \\ & & D_{n-2}(z) & & \\ & 0 & & \cos(t\frac{\pi}{2})\, D_{n-1}(z) & \sin(t\frac{\pi}{2})\, D_{n-1}(z)\, D_n(z) \\ & & & -\sin(t\frac{\pi}{2}) & \cos(t\frac{\pi}{2})\, D_n(z) \end{pmatrix}$$

Then $h^{(3)}(z,t)$ gives a homotopy deformation from

$$h^{(3)}(z,0) = D(z) \quad \text{to} \quad h^{(4)} = h^{(3)}(z,1) = \begin{pmatrix} D_1 & & & \\ & \ddots & & 0 \\ & & D_{n-2} & \\ & 0 & 0 & D_{n-1} D_n \\ & & -1 & 0 \end{pmatrix}$$

But then

$$h^{(4)}(z,t) = \begin{pmatrix} D_1 & & & \\ & \ddots & 0 & \\ 0 & & D_{n-2} & \\ & & D_{n-1} D_n \sin t\pi/2 & D_{n-1} D_n \cos t\frac{\pi}{2} \\ & & -\cos t\frac{\pi}{2} & \sin t\frac{\pi}{2} \end{pmatrix}$$

gives a homotopy deformation from $h^{(4)}$ to

$$\operatorname{diag}(D_1, \dots, D_{n-2}, D_{n-1} D_n, 1)$$

Continuing in this way, we see that h is homotopy

equivalent to $\operatorname{diag}(D_1 D_2 \cdots D_n, 1, \dots, 1) = \operatorname{diag}(\det D, 1, \dots, 1)$

Step 6

Write $\det D(z) = e^{iN\theta} D_0(z)$, $z = e^{i\theta}$

where $N = w(h) = w(D)$ and $w(D_0) = 0$. Set

$$D(z,t) = e^{iN\theta}\, e^{(1-t)\log D_0(z)}$$

As, it follows that $\log D_0(s)$ is continuous on S^1, and hence $\mathrm{diag}(D(z,t), 1, \ldots, 1)$ provides a homotopy deformation from $\mathrm{diag}(D(z), 1, \ldots, 1)$ to $\mathrm{diag}(e^{iN\theta}, 1, \ldots, 1)$

Combining the above steps we obtain the desired homotopy $h(z,t)$, $0 \le t \le 1$, taking $h(z)$ to $\mathrm{diag}(e^{iN\theta}, 1, \ldots, 1)$, proving Lemma 138.1. □

Proof of Proposition 137.2:

Define $T^{\#} : H_+ \to H_+$ by

$$T^{\#} f = P_+(h^{-1} f) \quad , \quad f \in H_+$$

Then for $f \in H_+$

$$T^{\#} T f = P_+ h^{-1} P_+ h f = f - P_+ h^{-1} P_- h f$$

We show that

$$K f = - P_+ h^{-1} P_- h f$$

is compact on H_+

For this purpose it is enough to show that $P_+ h^{-1} P_-$ is compact in $H^{(n)}$. Moreover as the compact operators are closed under norm convergence and finite linear combinations, it is sufficent by the Stone-Weierstass Theorem, to consider the case when $h^{-1}(z) = z^k E$ for some $k \in \mathbb{Z}$ and some constant matrix E. But then for $f = \sum_{-\infty}^{\infty} f_j z^j$

$$P_+ h^{-1} P_- f = P_+ \left(z^k E \sum_{-\infty}^{-1} f_j z^j \right)$$

$$= \chi_k \left(\sum_{j=-k}^{-1} f_j z^{j+k} \right)$$

where $\chi_k = 0$ if $k \leq 0$ and $\chi_k = 1$ if $k \geq 1$

Thus $P_+ h^{-1} P_-$ is finite rank and hence compact. In a similar way, we see that $T T^\# = 1 + \text{compact}$. Thus

$T^{\#}$ is a pseudo-inverse for T and hence T is Fredholm by Theorem 112.1.

Under the deformation $h(z,t)$ in Lemma 138.1 taking $h(z,0) = h(z)$ to

$$h(z,1) = \mathrm{diag}\,(e^{iN\theta}, 1, \dots, 1), \quad N = w(h),$$

$h(\cdot,t)$ is continuous and invertible on S^1 for all $0 \le t \le 1$, and hence $T(h(\cdot,t))$, $0 \le t \le 1$, is Fredholm by the preceding calculation. It follows then from Theorem 114+.1 that

(145.1) $$\mathrm{ind}\, T = \mathrm{ind}\, T_0$$

wher $T_0 = T(h_0)$, $h_0 = \mathrm{diag}\,(e^{iN\theta}, 1, \dots, 1)$

Suppose $N \ge 0$. Now if $T_0 f = 0$, $f = (f_1, \dots, f_n^T)$ $\in H_+$, then $P_+ e^{iN\theta} f_1 = e^{iN\theta} f_1 = 0$ and $P_+ f_i = f_i = 0$, $2 \le i \le N$. Hence $f = 0$ and so $\ker T_0 = \{0\}$. On the other hand, under the non-degenerate pairing

$$\langle g, f\rangle = \int_{S^1} g^T(z) f(z)\, dz \quad , \quad f, g \in H^{(n)}$$

(in otherwords, any linear ~~functiun~~ functional l on $H^{(n)}$ can be expressed uniquely by an element $g = g_l \in H^{(n)}$ and

$$l(f) = \langle g, f\rangle \quad)$$

we may identify H_+' with H_-. Indeed if $l \in H_+'$, then $Lf = l \circ P_+ f$ is in $H^{(n)'}$ and so

$$Lf = \langle g, f\rangle$$

for some $g \in H^{(n)}$. But then for $f \in H_+$

$$l(f) = Lf = \int_{S^1} g^T f\, dz$$

$$= \int_{S^1} \left((P_- g)^T + (P_+ g)^T\right) f\, dz$$

$$= \int g_-^T f\, dz$$

as $\int (P_+ g)^T f\, dz = 0$. As g_- is clearly unique, we see that we may identify H_+' with H_-. Also for $g \in H_+' \cong H_-$ and $f \in H_+$, $\langle T_0' g, f\rangle = \langle g, T_0 f\rangle = \langle g, P_+ h_0 f\rangle = \langle g, h_0 f\rangle$

$$= \int (g)^T h_0 f\, dz = \int (h_0^T g)^T f\, dz$$

$$= \int (P_- h_0^T g)^T f\, dz .$$

Thus for $g \in \mathcal{H}_- = \mathcal{H}_+'$,

(147.1) $$T_0' g = P_- h_0^T g$$

Now as T_0' is Fredholm, $\operatorname{codim} T_0 = \dim \ker T_0'$. If $T_0' g = 0$, $g = (g_1, \dots, g_n)^T \in \mathcal{H}_-$, Then we must have

$$P_- e^{iN\theta} g_1 = 0 \quad \text{and} \quad g_j = 0 \text{ for } 2 \le j \le n,$$

Writing $g_1 = \sum_{-\infty}^{-1} \gamma_j e^{ij\theta}$ we find $\gamma_j = 0$ for $j + N < 0$ i.e. $j < -N$. Thus $T_0' g = 0$ if and only if

$$g = \sum_{j=-N}^{-1} \gamma_j e^{ij\theta}$$

with $\gamma_{-1}, \dots, \gamma_{-N}$ arbitrary. Thus $\operatorname{codim} T_0 = \dim \ker T_0' = N$. It follows that for $N \ge 0$, $\operatorname{ind} T = \operatorname{ind} T_0 = 0 - N = -N = -\omega(h)$.

On the other hand, suppose $N < 0$. Then for $T^\#$ as above, we have by (120.2), $\operatorname{ind} T = -\operatorname{ind} T^{\#}$. As $w(h^{-1}) = -w(h) = -N > 0$ we conclude that $\operatorname{ind} T^{\#} = -w(h^{-1}) = w(h)$. Thus again $\operatorname{ind} T = -w(h)$, which completes the proof of Proposition 137.2 $\square$

As the final topic in this course we want to define the class $B_1(\mathcal{H})$ of <u>trace class</u> operators in a separable Hilbert space $\mathcal{H}$. For such operators $T \in B_1(\mathcal{H})$, we can define

$$\det(1 + T)$$

in an unambiguous way. The operators $B_1(\mathcal{H})$ thus identify the class of (infinite dimensional) operators for which "all" of the notions of finite dimensional linear algebra go through.