we can conclude that

$$\sum_{n=1}^{N} (\varphi_n, |A+B| \varphi_n) \le \mathrm{tr}|A| + \mathrm{tr}|B| < \infty$$

and thus $A+B \in \mathcal{B}_1$. But to prove (161.1), we need only to prove that

$$\mathrm{tr}\, U^* V |A| V^* U \le \mathrm{tr}|A|.$$

Picking an orthonormal basis $\{\varphi_n\}$ with each φ_n in $\ker U$ or in $(\ker U)^\perp$ we see that

$$\mathrm{tr}\, U^* V |A| V^* U = \sum (\varphi_n, U^* V |A| V^* U \varphi_n)$$

$$= \sum (U\varphi_n, V|A|V^* U\varphi_n)$$

$$\le \mathrm{tr}(V|A|V^*)$$

Similarly picking an orthonormal basis $\{\psi_m\}$ with each ψ_m in $\ker V^*$ or $(\ker V^*)^\perp$ we find $\mathrm{tr}\, V|A|V^*$ $\le \mathrm{tr}|A|$.

Lecture 13 (b) By the lemma proven below, each $B \in \mathcal{L}(\mathcal{H})$ can be written as a linear combination of 4 unitary operators and so by (a) we only need to show that $A \in \mathcal{B}_1$

$\Rightarrow$ $UA \in \mathcal{B}_1$ and $AU \in \mathcal{B}_1$ if U is unitary.

But $|UA| = \sqrt{(UA)^*UA} = \sqrt{A^*U^*UA} = \sqrt{A^*A} = |A|$

and $(U^{-1}|A|U)(U^{-1}|A|U) = U^{-1}|A|^2 U$
$= U^{-1}A^*AU$
$= (AU)^*AU = |AU|^2$

and so $|AU| = U^{-1}|A|U$ and so by part (c) of

Th^m 158.2, AU and $UA \in \mathcal{B}_1$.

(c) Let $A = U|A|$ and $A^* = V|A^*|$ be the polar

decompositions of A & A^*. Then $A^* = |A|U^*$

and $|A^*| = V^*V|A^*| = V^*A^* = V^*|A|U^*$

If $A \in \mathcal{B}_1$ then $|A| \in \mathcal{B}_1$ and so $|A^*| \in \mathcal{B}_1$, by (b),

and so $A^* = V|A^*| \in \mathcal{B}_1$, again by (b). $\square$

<u>Lemma 163.1</u> Every $B \in \mathcal{L}(H)$ can be written as a linear combination of 4 unitary operators

<u>Proof:</u> Since $B = \frac{1}{2}(B+B^*) - \frac{i}{2}(i(B-B^*))$, B can be

written as a linear combination of 2 self-adjoint operators. So, suppose A is self-adjoint and without loss we can assume that $\|A\| \leq 1$. Then

$$A \pm i\sqrt{1-A^2}$$

are unitary and $A = \frac{1}{2}\left(A + i\sqrt{1-A^2}\right) + \frac{1}{2}\left(A - i\sqrt{1-A^2}\right)$. $\square$

Thm 164.1

Let $\|\cdot\|_1$ be defined in $\mathcal{B}_1$ by $\|A\|_1 = \operatorname{tr}|A|$

Then $\mathcal{B}_1$ is a Banach space with norm $\|\cdot\|_1$ and

$\|A\| \leq \|A\|_1$.

Proof: Exercise.

Exercise 164.2

Show that if $A \in \mathcal{B}_1(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$ then

$$\|BA\|_1 \leq \|B\| \, \|A\|_1$$

$$\|AB\|_1 \leq \|B\| \, \|A\|_1$$

(Hint: Use min-max principle).

Th^{m} 165.1

Every $A \in \mathcal{B}_1$ is compact. A compact operator A is in $\mathcal{B}_1$ if and only if $\sum_1^\infty \lambda_n < \infty$ where λ_n are the singular values of A

Proof: Since $A \in \mathcal{B}_1$, $|A|^2 = A^*A \in \mathcal{B}_1$ and so

$$\operatorname{tr} |A|^2 = \sum_{n=1}^\infty \|A\varphi_n\|^2 < \infty$$

for any orthonormal basis $\{\varphi_n\}_{n=1}^\infty$. Suppose $\psi \in \langle \varphi_1, \dots, \varphi_N \rangle^\perp$ and $\|\psi\| = 1$, then we have

$$\|A\psi\|^2 \leq \operatorname{tr} |A|^2 - \sum_{n=1}^N \|A\varphi_n\|^2$$

since $\{\varphi_1, \dots, \varphi_N, \psi\}$ can always be completed to an orthonormal basis. Thus as $N \to \infty$

$$\sup \{ \|A\psi\| : \psi \in \langle \varphi_1, \dots, \varphi_N \rangle^\perp, \|\psi\| = 1 \} \to 0$$

Thus

$$\left(A - \sum_{n=1}^N (\varphi_n, \cdot) A\varphi_n \right) \psi = 0 \quad \text{if } \psi \in \langle \varphi_1, \dots, \varphi_N \rangle$$
$$= A\psi \quad \text{if } \psi \in \langle \varphi_1, \dots, \varphi_N \rangle^\perp.$$

Hence $\sum_{n=1}^N (\varphi_n, \cdot) A\varphi_n \to A$ in norm and so A is compact.

Finally as

(166.1) $$|A| = \sum \lambda_n (\varphi_n, \cdot) \varphi_n.$$

where ~~where~~ λ_n are the singular values of $|A|$, and

$|A|^2 \varphi_n = A^* A \varphi_n = \lambda_n^2 \varphi_n$, $\lambda_n \neq 0$, we have

$$\operatorname{tr} |A| = \sum (\varphi_n, |A| \varphi_n) = \sum \lambda_n$$

as we can always include the eigenvectors of $|A|$ corresponding to $\lambda_n = 0$ in (166.1).

<u>Corollary 166.2</u>

The finite rank operators are $\|\cdot\|_1$-dense in B_1.

The second class of operators which we will discuss are the <u>Hilbert - Schmidt operators</u> on a separable Hilbert space $\mathcal{H}$. This is a class of operators in $\mathcal{L}(\mathcal{H})$ is itself a Hilbert space.

<u>Definition 166.3.</u> An operator $T \in \mathcal{L}(\mathcal{H})$ is called <u>Hilbert-Schmidt</u> if and only if $\operatorname{tr} T^* T = \operatorname{tr} |T|^2 < \infty$.

The family of all Hilbert-Schmidt operators is denoted by B_2.

Th^{m} 167.1

(a) $\mathcal{B}_2$ is a $*$-ideal

(b) If $A, B \in \mathcal{B}_2$, Then for any orthonormal basis

$$\sum_{n=1}^{\infty} (e_n, A^* B e_n)$$

is absolutely summable, and its limit, denoted by $(A, B)_2$

is independent of the orthonormal basis chosen

(c) $\mathcal{B}_2$ with inner product $(A, B)_2$ is a Hilbert space

(d) If $\|A\|_2 = \sqrt{(A,A)_2} = (\operatorname{tr} A^* A)^{\frac{1}{2}}$ then

$$\|A\| \leq \|A\|_2 \leq \|A\|_1$$

and

$$\|A\|_2 = \|A^*\|_2$$

(e) Every $A \in \mathcal{B}_2$ is compact and a compact operator, A, is in $\mathcal{B}_2$ if and only if $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ where λ_n are the singular values of A.

(f) The finite rank operators are $\|\cdot\|_2$-dense in $\mathcal{B}_2$.

(g) $A \in \mathcal{B}_2$ if and only if $\{ \|A\varphi_n\| \in l_2$ for some orthonormal basis $\{\varphi_n\}\}$.

(***) (h) $A \in \mathcal{B}_1$ if and only if $A = BC$ with B, C in $\mathcal{B}_2$.

Proof: Exercise. Use arguments analogous to the case $\mathcal{B}_1$. □

Note that at the technical level it is much easier to detect when an operator T is in $\mathcal{B}_2$ rather than $\mathcal{B}_1$. All we have to do is show that

(168.1) $$\sum_{n=1}^{\infty} \|T\varphi_n\|^2 < \infty$$

for some orthonormal basis φ_n. The key point is that (168.1) requires a calculation involving only the operator T, which is given to us. However, to see if $T \in \mathcal{B}_1$, we must show that

(168.2) $$\sum_{n=1}^{\infty} (\varphi_n, |T| \varphi_n) < \infty$$

for some orthonormal basis φ_n. Here we are making a with $|T|$ which is a transcendental function of T:

$|T| = \sqrt{T^*T}$, and not at all explicit.

This is why (h) above is so useful: to show that $T \in \mathcal{B}_1$ we just to show that T is a product of 2 op's in $\mathcal{B}_2$, which is a non-transcendental task.

Moreover, if $\mathcal{H} = L^2(M, d\mu)$ for some measure space $(M, \mathcal{A}, d\mu)$, then $\mathcal{B}_2$ has a concrete realization

Thm 169.1 Let $(M, \mathcal{A}, d\mu)$ be a measure space and let $\mathcal{H} = L^2(M, d\mu)$ be separable. Then $A \in \mathcal{L}(\mathcal{H})$ is Hilbert-Schmidt if and only if there is a function

(169.2) $$K \in L^2(M \times M, d\mu \otimes d\mu)$$

with

(169.3) $$Af(x) = \int_M K(x,y)\, f(y)\, d\mu(y), \qquad f \in L^2,$$

Moreover,

(169.4) $$\|A\|_2^2 = \int |K(x,y)|^2\, d\mu(x)\, d\mu(y).$$

Proof: Let $K \in L^2(M \times M, d\mu \otimes d\mu)$ and let A_K be the associated integral operator. It is easy to see (exercise!) that A_K is a well-defined operator on $\mathcal{H}$ and that

(170.1) $$\|A\|_{K} \leq \|K\|_{L^2}$$

Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal basis for $L^2(M, d\mu)$. Then $\{\varphi_n(x)\overline{\varphi_m(y)}\}_{n,m=1}^{\infty}$ is an orthonormal basis for $L^2(M \times M, d\mu \otimes d\mu)$. (check this!) so

(170.2) $$K = \sum \alpha_{nm} \varphi_n(x) \overline{\varphi_m(y)}$$

Let
$$K_N = \sum_{n,m=1}^{N} \alpha_{nm} \varphi_n(x) \overline{\varphi_m(y)} .$$

Then each K_N is the integral kernel of a finite rank operator. In fact
$$A_{K_N} = \sum_{n,m=1}^{N} \alpha_{nm} (\varphi_m, \cdot) \varphi_n$$

Since $\|K_N - K\|_{L^2} \to 0$, $\|A_K - A_{K_N}\| \to 0$ as $N \to \infty$ by (170.1). Thus A_K is compact and in fact

$$\mathrm{tr}\, A_K^* A_K = \sum_{n=1}^{\infty} \|A_K \varphi_n\|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\alpha_{nm}|^2 = \|K\|_{L^2}$$

Thus $A_K \in \mathcal{B}_2$ and $\|A_K\|_2 = \|K\|_{L^2}$.

We have shown that the map K is an isometry of $L^2(M \times M, d\mu \otimes d\mu)$ into $\mathcal{B}_2$, so its range is closed. But the finite rank operators clearly come from kernels and since they are dense in $\mathcal{B}_2$ the range of $K \mapsto A_K$ is all of $\mathcal{B}_2$. $\square$

Note that if an operator A has a kernel K, then $\|K\|_{L^2} < \infty$ is a very useful condition to verify that that A is compact.

We now, finally define the trace of an operator.

<u>Th^m 171.1</u> If $A \in \mathcal{B}_1$ and $\{\varphi_n\}_{n=1}^{\infty}$ is any orthonormal basis, then $\sum_{n=1}^{\infty} (\varphi_n, A\varphi_n)$ converges absolutely and the limit is independent of the choice of basis.

<u>Proof:</u> We write $A = U|A|^{\frac{1}{2}}|A|^{\frac{1}{2}}$. Then

$$|(\varphi_n, A\varphi_n)| \le \| |A|^{\frac{1}{2}} U^* \varphi_n \| \, \| |A|^{\frac{1}{2}} \varphi_n \|$$

Thus

$$\sum_{n=1}^{\infty} |(\varphi_n, A\varphi_n)| \le \left(\sum_{n=1}^{\infty} \| |A|^{\frac{1}{2}} U^* \varphi_n \|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \| |A|^{\frac{1}{2}} \varphi_n \|^2 \right)^{\frac{1}{2}}$$

so since $|A|^{\frac{1}{2}} U^*$ and $|A|^{\frac{1}{2}}$ are in $\mathcal{B}_2$, the sum converges.

We have

(172.1) $$A = \sum_{n=1}^{\infty} \lambda_n (f_n, \cdot) g_n \qquad \text{when } \sum \lambda_n < \infty, \ \lambda_n > 0$$

$\{f_n\}$, $\{g_m\}$ orthonormal sets, by (153.1).

Thus for any orthonormal basis $\{\varphi_n\}$

$$\begin{aligned}
\sum_n (\varphi_n, A\varphi_n) &= \sum_n \sum_m \lambda_m (f_m, \varphi_n)(\varphi_n, g_m) \\
&= \sum_m \lambda_m \sum_n (f_m, \varphi_n)(\varphi_n, g_m) \\
&= \sum_m \lambda_m (f_m, g_m)
\end{aligned}$$

which is independent of $\{\varphi_n\}$. $\square$

<u>Definition 173.1</u> The map

$$\mathrm{tr} : \mathcal{B}_1 \to \mathbb{C}$$

given by $\mathrm{tr}\, A = \sum_{n=1}^{\infty} (\varphi_n, A\varphi_n)$ where $\{\varphi_n\}$ is any orthonormal basis is called the <u>trace</u>.

It is <u>not</u> true (<u>exercise</u>!) that $\sum_{n=1}^{\infty} |(\varphi_n, A\varphi_n)| < \infty$ for some orthonormal basis implies that $A \in \mathcal{B}_1$. For A to be in $\mathcal{B}_1$, it is necessary that the sum is finite for <u>all</u> orthonormal bases.

Here are properties of the trace.

<u>Th^m 173.1</u>

(a) $\mathrm{tr}(\cdot)$ is linear

(b) $\mathrm{tr}\, A^* = \overline{\mathrm{tr}\, A}$

(c) $\mathrm{tr}\, AB = \mathrm{tr}\, BA$ if $A \in \mathcal{B}_1$ and $B \in \mathcal{B}_2$

(d) $|\mathrm{tr}\, A| \leq \|A\|_1$

<u>Proof</u>: (a) and (b) are obvious. To prove (c), use

(172.1), $A = \sum \lambda_n (f_n, \cdot) g_n$

$$\operatorname{tr} AB = \operatorname{tr}\left(\sum \lambda_n (B^* f_n, \cdot) g_n \right)$$

$$= \sum_m \left(g_m, \left(\sum_n \lambda_n (B^* f_n, \cdot) g_n\right) g_m\right)$$

as the g_m's are orthonormal and if $g \perp \{g_m\}$ then

$(g, ABg) = 0$. Thus

$$\operatorname{tr} AB = \sum_n \lambda_m (B^* f_m, g_m) = \sum \lambda_m (f_m, Bg_m)$$

and

$$\operatorname{tr} BA = \operatorname{tr}\left(\sum_n \lambda_n (f_n, \cdot) Bg_n\right)$$

$$= \sum_m \left(f_m, \left(\sum \lambda_n (f_n, \cdot) Bg_n\right) f_m\right)$$

$$= \sum_m \lambda_m (f_m, Bg_m)$$

and so $\operatorname{tr} AB = \operatorname{tr} BA$. Finally

(d) $$|\operatorname{tr} A| = \left|\sum (\varphi_n, A\varphi_n)\right|$$

$$= \left|\sum_n \left(\varphi_n, \left(\sum \lambda_m (f_m, \cdot) g_m\right) \varphi_n\right)\right|$$

Let $\{\varphi_n\}$ be the completion of $\{g_m\}$ to an orthonormal basis

Then $|\operatorname{tr} A| = \left|\sum \lambda_m (f_m, g_m)\right| \le \sum \lambda_m = \operatorname{tr} |A|$

as $|(f_m, g_m)| \le \|f_m\| \|g_m\| \le 1$. $\square$

The following result show that $\mathcal{B}_1$ is the dual space of the compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H})$ and the dual space of $\mathcal{B}_1$ is just $\mathcal{L}(\mathcal{H})$.

Th^m 175.1

(a) $\mathcal{B}_1 = (\mathcal{K}(\mathcal{H}))^*$, i.e., the map $A \mapsto \mathrm{tr}(A\,\cdot)$ is an isometric isomorphism of $\mathcal{B}_1$ onto $(\mathcal{K}(\mathcal{H}))^*$.

(b) $\mathcal{L}(\mathcal{H}) = (\mathcal{B}_1)^*$, i.e., the map $B \mapsto \mathrm{tr}(B\,\cdot)$ is an isometric isomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{B}_1^*$.

In particular $\mathcal{K}(\mathcal{H})$ is a non-refexive Banach space.

Proof: See Reed-Simon Vol I, Problem 30, Chapter VI. $\square$

~~We are now in a position to define the determinant of an appropriate class of operators, viz, $\det(1+A)$ where $A \in \mathcal{B}_1$.~~

<u>Comment</u>: We have introduced $\mathcal{B}_1$ and $\mathcal{B}_2$ and saw that for $A \in \mathcal{B}_1$, $\|A\|_1 = \sum \lambda_n$ and for $A \in \mathcal{B}_2$, $\|A\|_2 = (\sum \lambda_n^2)^{\frac{1}{2}}$, where the λ_n's are the singular values of A. For general $p \geq 1$, we define the p^{th} <u>Schatten-class $\mathcal{B}_p$</u> as the class of compact operators A with $\sum_{n=1}^{\infty} \lambda_n^p < \infty$:

$\mathcal{B}_p$ is a Banach space with norm $\|A\|_p = (\sum_1^{\infty} \lambda_n^p)^{\frac{1}{p}}$.

For more details, e.g., B. Simon, Trace ideals and their applications.

Just as

$$\operatorname{tr}|A| = \sum \lambda_n \quad , \text{ where the } \lambda_n\text{'s are}$$

the singular values of A and hence the eigenvalues of $|A|$, we expect that if $A \in \mathcal{B}_1$, then

(176.1) $$\operatorname{tr} A = \sum \sigma_n$$

where the σ_n's are the eigenvalues of A. This is true, but, as we will see, rather difficult to prove.

We are now in a position to define the determinant of an appropriate class of operators, viz.,

$$\det(I + A)$$

where $A \in \mathcal{B}_1$.

We need some <u>alternating algebra</u> i.e. the theory of antisymmetric tensor spaces.

Given a Hilbert space $\mathcal{H}$, $\bigotimes^n \mathcal{H}$ is defined as the vector space of multilinear functionals on $\mathcal{H}$. Explicitly, given $\varphi_1, \dots, \varphi_n \in \mathcal{H}$, we define $\varphi_1 \otimes \cdots \otimes \varphi_n \in \bigotimes^n \mathcal{H}$ by

$$(\varphi_1 \otimes \cdots \otimes \varphi_n)\langle \eta_1, \dots, \eta_n \rangle \equiv (\varphi_1, \eta_1) \cdots (\varphi_n, \eta_n)$$

for $\eta_i \in \mathcal{H}$, $1 \le i \le n$. It is an exercise (cf Reed Simon Vol I, p49 et seq) that the finite span of the $\{\varphi_1 \otimes \cdots \otimes \varphi_n\}$ possesses a well defined inner product with

$$(\varphi_1 \otimes \cdots \otimes \varphi_n, \eta_1 \otimes \cdots \otimes \eta_n)$$

$$= (\varphi_1, \eta_1) \cdots (\varphi_n, \eta_n)$$

$\otimes^n \mathcal{H}$ is the completion of this finite span in the topology generated by this inner product. Given any $A \in \mathcal{L}(\mathcal{H})$, there is a natural operator $T_n(A)$ in $\mathcal{L}(\otimes^n \mathcal{H})$ with

$$T_n(A)(\varphi_1 \otimes \cdots \otimes \varphi_n) = A\varphi_1 \otimes \cdots \otimes A\varphi_n$$

T_n satisfies

(178.1) $$T_n(AB) = T_n(A)\, T_n(B)$$

Let $\mathcal{P}_n$ denote the group of all permutations on n letters. Let $\varepsilon(\cdot)$ be the function on $\mathcal{P}_n$ that is $+1$ (resp. -1) on even (resp. odd) permutations. Define $\varphi_1 \wedge \cdots \wedge \varphi_n \in \otimes^n \mathcal{H}$ by

(178.2) $$\varphi_1 \wedge \cdots \wedge \varphi_n = \frac{1}{(n!)^{\frac{1}{2}}} \sum_{\pi \in \mathcal{P}_n} \varepsilon(\pi) [\varphi_{\pi(1)} \otimes \cdots \varphi_{\pi(n)}]$$

and define $\Lambda^n(\mathcal{H})$ to be the subspace of $\otimes^n \mathcal{H}$

spanned of $\{\varphi_1 \wedge \cdots \wedge \varphi_n\}$. The $(n!)^{-\frac{1}{2}}$ normalization factor is chosen so that if $\varphi_1, \ldots, \varphi_n$ are orthonormal, then $\varphi_1 \wedge \cdots \wedge \varphi_n$ has norm one. More generally, (exercise) one has

(179.1) $$(\varphi_1 \wedge \cdots \wedge \varphi_n, \eta_1 \wedge \cdots \wedge \eta_n) = \det(\varphi_i, \eta_j)_{1 \le i, j \le n}$$

Given $A \in \mathcal{L}(\mathcal{H})$, $\Gamma_n(A)$ leaves $\Lambda^n(\mathcal{H})$ invariant,

$$\Gamma_n(A)\, \varphi_1 \wedge \cdots \wedge \varphi_n = \sum_{\pi \in \mathcal{P}_n} \varepsilon(\pi)\, A\varphi_{\pi(1)} \otimes \cdots \otimes A\varphi_{\pi(n)}$$

$$= A\varphi_1 \wedge \cdots \wedge A\varphi_n$$

and we denote its restriction to $\Lambda^n(\mathcal{H})$ by $\Lambda^n(A)$.

We have from (178.1)

(179.2) $$\Lambda^n(AB) = \Lambda^n(A)\, \Lambda^n(B)$$

When $n = 0$, we define $\Lambda^n \mathcal{H}$ to be $\mathbb{C}$ and $\Lambda^n(A)$ as $I : \mathbb{C} \to \mathbb{C}$.

The connection between determinants of finite dimensional operators and $\Lambda^n(\cdot)$ is given by: