Lecture 14

Lemma 180.1

(a) Let $\mathcal{H}$ be an n-dimensional Hilbert space. Then $\Lambda^n \mathcal{H}$ is 1 dimensional

(b) If $\mathcal{H}$ has dimension n, then $\Lambda^n(A)$ is multiplication by the number $\det A$, the ordinary determinant of A.

c) $\det(AB) = \det A \det B$.

Proof: (a) We shall show more generally that for $0 \le k \le n$.

(180.2) $$\dim \Lambda^k(\mathcal{H}) = \binom{n}{k}.$$

For let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathcal{H}$.

We claim that $\{ e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n \}$

is an orthonormal basis for $\Lambda^k(\mathcal{H})$: This proves (180.2).

Direct calculation shows that these vectors are orthonormal, and hence independent. Moreover, they span $\Lambda^n(\mathcal{H})$, for if $u_i = \sum a_{ij} e_j$, then

(181.1) $\quad u_1 \wedge \dots \wedge u_n = \sum a_{1j_1} \cdots a_{nj_n} \, e_{j_1} \wedge e_{j_2} \cdots \wedge e_{j_n}$.

But it is clear from (178.2) that if $u_i = u_j$ for some $i \neq j$, then $u_1 \wedge \dots \wedge u_n = 0$. Thus the only terms that survive in the sum (181.1), are the vectors $e_{j_1} \wedge \dots \wedge e_{j_n}$ where the j_i's are distinct. Thus the vectors are a spanning set, as required.

(b) As $\Lambda^n(H)$ is 1-dimensional, $\Lambda^n(A)$ must be mult. by some number α. If $e_1, \dots, e_n$ is an orthonormal basis, then

$$\alpha = (e_1 \wedge \dots \wedge e_n, \Lambda^n(A)\, e_1 \wedge \dots \wedge e_n)$$

$$= (e_1 \wedge \dots \wedge e_n, Ae_1 \wedge \dots Ae_n)$$

$$= \det(e_i, Ae_j) = \det A, \quad \text{by (179.1).}$$

$$\cancel{= \sum A_{1j_1} A_{2j_2} \cdots A_{nj_n} (e_1 \wedge \dots \wedge e_n, e_{j_1} \wedge \dots \wedge e_{j_n})}$$

$$\cancel{= \sum A_{1j_1} \cdots A_{nj_n} \, \varepsilon(j) (e_1 \wedge \dots \wedge e_n, e_1 \wedge \dots \wedge e_n)}$$

(c) follows from (b) and (179.2). □

The rather effortless proof of $\det AB = \det A \det B$ shows the power of alternating algebra in studying determinants.

<u>Th^m 182.1</u> Let A be a compact operator in a separable Hilbert space $\mathcal{H}$. Then there exists an orthonormal set $\{e_n\}_{n=1}^{N(A)}$ so that

$$A e_n = \lambda_n(A) e_n + \sum_{m=1}^{n-1} \nu_{nm} e_m$$

for suitable ν_{nm}. In particular

$$(e_n, A e_n) = \lambda_n .$$

Here $\{\lambda_n(A)\}$ are the non-zero eigenvalues of A, counted with their algebraic multiplicity, ie $\dim \operatorname{ran} P_{\lambda_i}$ where

$$P_{\lambda_i} = \frac{1}{2\pi i} \oint_Q \frac{ds}{s-A}$$

and Q is a small circle such that $(\text{interior } Q) \cap \operatorname{spec} A = \{\lambda_i\}$.

<u>Proof</u>: Exercise.

<u>Note</u>: The orthonormal set $\{e_n(A)\}_{n=1}^{N(A)}$ is called a Schur basis (although it need not be a basis)

<u>Exercise</u> If A is an $n \times n$ matrix show that there exists a unitary matrix Q such that $Q^* A Q = U$ where U is upper triangular. U is known as the <u>Schur form</u> for A.

Suppose A is an $n\times n$ matrix. Then if $\{\lambda_i\}_{i=1}^{N(A)}$ are the eigenvalues of A counting multiplicities, and if $e_1,\ldots,e_n$ is a Schur basis for A, then

$$\det(1+A) = (e_1\wedge\cdots\wedge e_n,\ \Gamma^n(1+A)\, e_1\wedge\cdots\wedge e_n)$$

$$= (e_1\wedge\cdots\wedge e_n,\ (1+A)e_1\wedge\cdots\wedge(1+A)e_n)$$

$$= \prod_{i=1}^{n}(1+\lambda_i)$$

and

$$\operatorname{tr}(\Lambda^k(A)) = \sum_{1\le i_1<\cdots<i_k\le n}(e_{i_1}\wedge\cdots\wedge e_{i_k},\ \Lambda^k(A)\, e_{i_1}\wedge\cdots\wedge e_{i_k})$$

$$= \sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

Thus

$$\det(1+A) = \prod_{i=1}^{n}(1+\lambda_i) = 1+\sum_i\lambda_i+\sum_{i<j}\lambda_i\lambda_j+\sum_{i<j<k}\lambda_i\lambda_j\lambda_k+\cdots$$

i.e.,

(183.1) $$\det(1+A) = \sum_{j=0}^{n}\operatorname{tr}\Lambda^j(A)$$

in case $\dim\mathcal{H}=n$. In the case $\dim\mathcal{H}=\infty$, we shall <u>define</u> $\det(1+A)$ by (183.1)

Lemma 184.1 Let A be a trace class operator with singular values $\mu_n(A)$. Then for any k, $\Lambda^k(A)$ is a trace class operator and

(184.2) $$\|\Lambda^k(A)\|_1 = \sum_{1\le i_1<\cdots<i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A)$$

(184.3) $$\|\Lambda^k(A)\|_1 \le \|A\|_1^k / k!$$

Proof: Let $A = U|A|$ be the polar decomposition of A. Then it is easy to see that $\Lambda^k(A) = \Lambda^k(U)\Lambda^k(|A|)$ is the polar decomposition for $\Lambda^k(A)$ and in particular

$$|\Lambda^k(A)| = \Lambda^k(|A|)$$

Let $e_1, \ldots$ be an orthonormal basis of eigenvectors for $|A|$. Then $e_{i_1}\wedge\cdots\wedge e_{i_k}$ are an orthonormal basis of eigenvectors for $\Lambda^k(|A|)$. Thus

$$\operatorname{tr}|\Lambda^k(A)| = \sum_{i_1<\cdots<i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A)$$

$$\le \frac{1}{k!}\sum_{i_1,\ldots,i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A)$$

$$= \frac{(\operatorname{tr}(|A|))^k}{k!}.$$

Thus $\Lambda^k(A)$ is trace class and (184.2)(184.3) hold. ∎

__Definition 185.1__ Let $A \in \mathcal{B}_1$. Then

(185.2) $$\det(1+A) \equiv \sum_{k=0}^{n} \operatorname{tr}\left(\Lambda^k(A)\right).$$

__Lemma 185.3__ The sum (185.2) converges absolutely for each $A \in \mathcal{B}_1$. Moreover

(a) $$|\det(1+A)| \le \prod_{j=1}^{\infty} (1+\mu_j(A))$$

(b) $$|\det(1+A)| \le \exp(\|A\|_1)$$

(c) For any $A_1, \dots, A_n \in \mathcal{B}_1$,

$$\langle z_1, \dots, z_n\rangle \mapsto \det\left(1 + \sum_{j=1}^{n} z_j A_j\right)$$

is an entire analytic function.

(d) For any $A, B \in \mathcal{B}_1$,
$$|\det(1+A) - \det(1+B)| \le \|A-B\|_1 \exp(\|A\| + \|B\| + 1)$$

Proof: Since $|\mathrm{tr}(\Lambda^k(A))| \le \|\Lambda^k(A)\|_1$

$$\le \frac{(\mathrm{tr}|A|)^k}{k!} = \frac{\|A\|_1^k}{k!}$$

we see that (185.2) converges absolutely. Thus

$$|\det(1+A)| \le \sum_{k=0}^{\infty} \frac{\|A\|_1^k}{k!} = e^{\|A\|_1}$$, which proves (b)

Also

$$|\det(1+A)| \le \sum_{k=0}^{\infty} \|\Lambda^k(A)\|_1$$

$$= \sum_{k=0}^{\infty} \sum_{i_1<\cdots<i_k} \mu_{i_1}(A)\cdots\mu_{i_k}(A)$$

$$= 1 + \sum \mu_{i_1}(A) + \sum_{i_1<i_2} \mu_{i_1}\mu_{i_2} + \cdots$$

$$= \prod_{j=1}^{\infty} (1+\mu_j(A))$$

which proves (a). In view of our estimates on the terms in (185.2) its enough to show that $\langle z_1, \ldots, z_n\rangle \to \mathrm{tr}(\Lambda^k(z_1A_1 + \cdots + z_nA_n))$ is analytic. But clearly $\mathrm{tr}(\Lambda^k(z_1A_1 + \cdots + z_nA_n))$ is a $\mathcal{B}_1$-valued analytic function and so its trace is is analytic as $tr(\cdot)$ is a bded linear functional on $\mathcal{B}_1$. This proves (c): (d) follows from (b) & (c) and the following Lemma. $\square$.

Lemma 187.1

Let X be a complex Banach space. Let $F: X \to \mathbb{C}$ be a function with the following properties.

(i) For any $x, y \in X$, $\mu \mapsto F(x+\mu y)$ is an entire function of μ.

(ii) For some monotone increasing function G on $[0, \infty)$,

$$|F(x)| \le G(\|x\|)$$

$\forall x \in X$

Then

$$|F(x) - F(y)| \le \|x-y\| \, G(\|x\| + \|y\| + 1).$$

Proof: See Reed-Simon IV pp 324-325. $\square$

We will eventually prove, in analogy with the matrix case, that for $A \in \mathcal{B}_1$,

(188.1) $$\det(1+A) = \prod_{n=1}^{\infty}(1+\lambda_n(A))$$

and

(188.2) $$\operatorname{tr} A = \sum_{n=1}^{\infty} \lambda_n(A) \qquad (\underline{\text{Lidskii's Th}^{\text{m}}})$$

where $\{\lambda_n(A)\}$ are the eigenvalues of A, counting multiplicity.

The strategy in proving (188.2) is to prove (188.1) first. Then replacing $A \to zA$ we have

$$\det(1+zA) = \prod_{n=1}^{\infty}(1+z\lambda_n(A))$$

Then the linear term in z on the LHS is $\operatorname{tr} A$ and on the RHS, is $\sum \lambda_n(A)$, which establishes (188.2)

First we summarize some properties of $\det(1+A)$

<u>Th$^{\text{m}}$ 188.3</u> Let A and $B \in \mathcal{B}_1$. Then

(a) $\det(1+A)\det(1+B) = \det(1+A+B+AB)$

(b) $1+A$ is invertible iff $\det(1+A) \neq 0$

(c) If $-\mu^{-1}$ is an eigenvalue of A, then $\det(1+zA)$

has a zero of order n at $z = \mu$ where n is the algebraic multiplicity of $-\mu^{-1}$.

(d) For any ε, $\exists\, C_\varepsilon$, depending on A, so that

$$|\det(1 + zA)| \le C_\varepsilon \exp(\varepsilon |z|).$$

Proof:

(a) Exercise — enough to show (a) for finite rank A and B

(b) If $1+A$ is invertible then $(1+A)^{-1} = 1 + B$ where $B = -A(1+A)^{-1} \in \mathcal{B}_1$. Thus by (a)

$$\det(1+A)\det(1+B) = \det(1 + A + B + AB)$$
$$= \det 1 = 1$$

so $\det(1+A) \neq 0$. If $1+A$ is not invertible, then -1 is an eigenvalues of A, so $\det(1+A) = 0$ by the result (c) that we are about to prove.

(c) Let P be the spectral projection onto $-\mu^{-1}$.

$$P = \oint \frac{1}{s-A}\,\frac{ds}{2\pi i}$$

Let $B = AP$ and $C = A(1-P)$

Then

$$(1+zA) = (1+zB)(1+zC).$$

so

$$\det(1+zA) = \det(1+zB)(\det(1+zC))$$

Since $1+zC$ is invertible for z near $-\mu^{-1}$, it suffices to show that $\det(1+zB)$ has an n^{th} order zero at $-\mu^{-1}$, where $n = \dim \operatorname{ran} P$.

Let $e_1, \ldots, e_n$ be a Schur basis for B acting in $\operatorname{ran} P$. Then

(190.1) $$Be_i = \lambda_i e_i + \sum_{j=1}^{i-1} d_{ij} e_j, \quad 1 \le i \le n,$$

where $\lambda_1 = \cdots = \lambda_n = -\mu^{-1}$. Extend $\{e_1, \ldots, e_n\}$ to an orthonormal basis for $\mathcal{H}$: $e_1, \ldots, e_n, e_{n+1}, \ldots$.

Then if $i > n$, as $\operatorname{ran} B \subset \operatorname{ran} P$,

$$Ae_i = \sum_{j=1}^{n} d_{ji} e_j$$

Thus (190.1) holds for all $\{e_i\}$, with $\lambda_i = 0$ for $i > n$, and for suitable $\{d_{ji}\}$

It follows easily that $\operatorname{tr}(\Lambda^k(B)) = \binom{n}{k}(-\mu^{-1})^k$ for $k \le n$ and $\operatorname{tr}\Lambda^k(B) = 0$ for $k > n$. Thus

$$\det(1+zB) = \sum_{k=0}^{n} \binom{n}{k}(-\mu^{-1})^k z^k = (1-z\mu^{-1})^n .$$

Thus $\det(1+zB)$ has an n^{th} order pole at $z = \mu$.

(d) Let $\mu_n(A)$ be the singular values of A. Choose N st $\sum_{n>N} \mu_n(A) < \varepsilon/2$. Then 185.3(a)

$$|\det(1+zA)| \le \prod_{j=1}^{\infty} (1+|z|\mu_j(A))$$

$$\le \left(\prod_{j=1}^{N} (1+|z|\mu_j(A))\right) e^{\frac{\varepsilon}{2}|z|}$$

since $1+x \le e^x$ for $x \ge 0$. Now since $\prod_{j=1}^{N}(1+|z|\mu_j(A))$ is a polynomial in $|z|$, we can find C_ε with

$$\prod_{j=1}^{N}(1+\mu_j(A)|z|) \le C_\varepsilon e^{\frac{\varepsilon}{2}|z|} . \qquad \blacksquare$$

In order to prove (188.1), we need the following result of Borel - Caratheodory.

Lemma 192.1 (Borel – Caratheodory)

Let f be analytic in a nbhood of $|z| < R$.

Then for any $r < R$,

$$\max_{|z| \leq r} |f(z)| \leq \frac{2r}{R-r} \max_{|z|=R} \left[\mathrm{Re}\,(f(z)) \right] + \frac{R+r}{R-r} |f(0)|$$

Proof: See e.g. RS Vol IV, pp 328-329. ∎

Finally we have

Th^m 192.2 For any $A \in \mathcal{B}_1$,

$$\det(1+A) = \prod_{j=1}^{N(A)} (1 + \lambda_j(A))$$

where $\{\lambda_j(A)\}_{j=1}^{N(A)}$ are the eigenvalues of A counted with algebraic multiplicity.

Proof: (cf Hadamard's general theorem on the expansion of an entire function in terms of its zero's: here there are no convergence factors or prefactors.)

Let $f(z) = \det(I + zA)$. Let

$$g(z) = \prod_{j=1}^{N(A)} (1 + z\lambda_j(A))$$

If $N(A) = \infty$, the product in question converges to an analytic function by standard results as.

(193.1) $$\sum_{j=1}^{\infty} |\lambda_j| \leq \sum \mu_j$$

where μ_i are the singular values of A. Indeed if $A = \sum_{n=1}^{\infty} \mu_n (f_n, \cdot) g_n$ is the canonical expansion of A with $\{f_n\}, \{g_n\}$ orthonormal sets, then if $\{e_i\}$ is a Schur basis for A

$$\lambda_m(A) = (e_m, Ae_m) = \sum \mu_n a_{nm}$$

where $a_{nm} = (f_n, e_m)(e_m, g_n)$. Thus

$$\sum_m |\lambda_m(A)| \leq \sum_n \mu_n \sum_m |(f_n, e_m)| \, |(e_m, g_n)|$$

$$\leq \sum_n \mu_n \left(\sum_m |(f_n, e_m)|^2\right)^{\frac{1}{2}} \left(\sum_m |(e_m, g_n)|^2\right)^{\frac{1}{2}}$$

$$= \sum_n \mu_n \|f_n\| \|g_n\| = \sum \mu_n$$

(Remark: (193.1) is a special case of the Schur - Lalesco - Weyl thm which states that for any $1 \le p < \infty$

$$\sum_{j=1}^{N(A)} |\lambda_j(A)|^p \le \sum_{j=1}^{\infty} \mu_j(A)^p \quad .)$$

We will show that $f(z) = g(z)$. By (b) (c) of thm 188.3, f and g have the same zeros including order, so f/g is an entire non-vanishing analytic function. Thus $f = g e^h$ where the ambiguity is determined by requiring that h be entire with $h(0) = 0$. This is possible as $f(0) = g(0) = 1$. We will show that $h(z) = 0$ for $|z| < 1$, so $h(z) \equiv 0$. For $R > 2$, and $|z| < R$ define

$$h_R(z) = \ln f_R(z)$$

$$k_R(z) = - \sum_{\{j : |\lambda_j|^{-1} > R\}} \ln(1 + z\lambda_j(A))$$

$$f_R(z) = f(z) \Big/ \prod_{\{j : |\lambda_j|^{-1} \le R\}} (1 + \lambda_j z)$$

The ambiguities in h_R and k_R are determined as analytic functions in $|z|<R$ by setting $h_R(0) = k_R(0) = 0$. Note also that $f_R(z)$ is an entire function. Since $h = h_R + k_R$, it suffices to show that $|h_R(z)| \to 0$ and $|k_R(z)| \to 0$ as $R \to \infty$ for each z with $|z| \le 1$.

Since $\ln(1+x)$ vanishes at $x = 0$ and is analytic in a nbhood of $\{x : |x| \le \frac{1}{2}\}$, we have

$$|\ln(1+x)| \le c\,|x|$$

for suitable c and all x with $|x| \le \frac{1}{2}$. Thus for $R \ge c$, and $|z| \le 1$,

$$|k_R(z)| \le |z| \sum_{\{j : |\lambda_j|^{-1} > R\}} |\lambda_j(A)| \to 0$$

as $R \to \infty$, since the infinite sum is convergent.

Next consider the entire function $f_R(z)$. If $|z| = 2R$ and $|\lambda_j|^{-1} \le R$, then $|1 + z\lambda_j| \ge 1$.

Thus if $|z| = 2R$, $\displaystyle |f_R(z)| \le \frac{|\det(1+zA)|}{\prod_{\{j : |\lambda_j|^{-1} \ge R\}} |(1+z\lambda_j)|}$

$$\underset{\text{by } 188.3(d)}{\le} \quad C_\varepsilon \frac{e^{\varepsilon |z|}}{1} = C_\varepsilon e^{2\varepsilon R}.$$

By the max. modulus principle, this remains true if

$|z| = R$, so

(196.1) $$\operatorname{Re} h_R(z) \le \ln C_\varepsilon + 2\varepsilon R.$$

for $|z| = R$. But by Lemma 192.1 above

$$\max_{|z| \le 1} |h_R(z)| \le \frac{2}{R-1} \max_{|z| = R} \operatorname{Re} h_R(z) + \frac{R+1}{R-1} h_R(0)$$

$$\underset{\substack{h_R(0) = 0 \\ \text{and } (196.1)}}{=} \frac{2}{R-1}\left(\ln C_\varepsilon + 2\varepsilon R\right)$$

Thus for any $|z| \le 1$

$$\overline{\lim_{R \to \infty}} |h_R(z)| \le 4\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $|h_R(z)| \to 0$ as $R \to \infty$. This proves the Thm. $\blacksquare$

<u>Corollary 197.1</u> (Lidskii's Th^m)

For any $A \in \mathcal{B}_1$,

$$\operatorname{tr} A = \sum_{j=1}^{N(A)} \lambda_j(A)$$

where $\lambda_j(A)$ are the eigenvalues of A, counting algebraic multiplicity.

<u>Proof</u>: Compare the terms linear in z from the identity in Th^m 192.2

$$\det(1+zA) = \prod_{j=1}^{N(A)} (1+z\lambda_j(A)) \quad . \quad \blacksquare$$