On the other hand, by (24.3),

$$\|Tx\| = \sup_{\|y'\| \le 1} |y'(Tx)| = \sup_{\|y'\| \le 1} |T'y'(x)|$$

$$\le \sup_{\|y'\| \le 1} \|T'y'\| \|x\|$$

$$\le \sup_{\|y'\| \le 1} \|T'\| \|y'\| \|x\| = \|T'\| \|x\|$$

and so $\|T\| \le \|T'\|$. Thus we have equality in (40.2)

(41.1) $$\|T'\| = \|T\|$$

Moreover the following is true:

Lecture 4

**Theorem 41.2** Let $T \in \mathcal{L}(X, Y)$. Then $T$ is a bijection from $X \to Y$ if and only if $T'$ is a bijection from $Y' \to X'$ and if $T$, or equivalently $T'$, is a bijection, then $(T')^{-1} = (T^{-1})'$ and as $T^{-1} \in \mathcal{L}(Y, X)$, $(T')^{-1} \in \mathcal{L}(X', Y')$.

**Remark:** In finite dimensions, this result is reflected in the fact that $\det A = \det A^T$ for any square matrix $A$.

**Proof:** Indeed if $T$ is a bijection, then unravelling the definition we see immediately that

(41.3) $$\langle (T^{-1})' x', y \rangle = \langle x', T^{-1} y \rangle \quad \forall x' \in X', y \in Y$$

implies

$$\langle (T^{-1})'x', Tx\rangle = \langle x', T^{-1}(Tx)\rangle = x'(x), \quad \forall x \in X, \; x' \in X'.$$

$\Rightarrow$ $$\langle T'(T^{-1})'x', x\rangle = \langle x', x\rangle \quad \forall x$$

ie $$T'(T^{-1})'(x') = x'$$

and also for $x' = T'y'$, (41.3) $\Rightarrow$

$$\langle (T^{-1})'T'y', y\rangle = \langle T'y', T^{-1}y\rangle$$

$$= \langle y', T(T^{-1}y)\rangle, \quad \forall y \in Y$$

$$= \langle y', y\rangle \quad \forall y \in Y$$

$\Rightarrow$ $$(T^{-1})'T'y' = y'$$

Hence $T'$ is a bijection and $(T')^{-1} = (T^{-1})'$. Conversely suppose that $T'$ is a bijection. By (24.1), given any $0 \neq x \in X$, choose $x' \in X'$ st $x'(x) = \|x\|$ and $\|x'\| = 1$. As $T'$ is a bijection, $\exists\, y' \in Y'$ st $x' = T'y'$. Hence

(42.1) $$\|x\| = x'(x) = \langle x', x\rangle = \langle T'y', x\rangle = \langle y', Tx\rangle$$

and so $Tx = 0 \Rightarrow x = 0$. We conclude that $T$ is injective. On the other hand, (42.1) shows that for any $x$,

$$\|x\| \leq \|y'\| \|Tx\|$$

But by (28.2), for some $c > 0$, $\|y'\| \leq \frac{1}{c}\|T'y'\| = \frac{1}{c}\|x'\| = \frac{1}{c}$

Thus

$$\|x\| \leq \frac{1}{c}\|Tx\|$$

and we conclude, again, by (28.2), that ran $T$ is closed.

Suppose $\operatorname{ran} T = \overline{\operatorname{ran} T} \subsetneq Y$ and let $0 \neq y_0 \in Y \setminus \operatorname{ran} T$. Then by Proposition 26.1 there exists $y' \in Y'$ such that $y'(y_0) = \|y_0\|$ and $y'(Tx) = 0 \ \forall x \in X$. But then for all $x \in X$,

$$\langle T'y', x\rangle = \langle y', Tx\rangle = 0$$

and so $T'y' = 0$. But $y'(y_0) = \|y_0\| \neq 0$, and this contradicts the fact that $\ker T' = \{0\}$. Thus $T$ is a bijection and $(T')^{-1} = (T^{-1})'$. $\square$

Exercise: If $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$, then $(ST)' \in \mathcal{L}(Z',X')$ and $(ST)' = T' \circ S'$. $\square$

Let $(\mathcal{H}, (\cdot,\cdot))$ be a Hilbert Space and let $T \in \mathcal{L}(\mathcal{H})$. Then for any $x \in \mathcal{H}$, the map $y \mapsto (x, Ty)$ defines a bounded linear functional on $\mathcal{H}$ and hence by the Riesz representation Theorem, there exists a unique $x^* \in \mathcal{H}$ st

(43.1) $$(x, Ty) = (x^*, y) \qquad \forall y \in \mathcal{H}$$

Set

(44.1) $$T^*x = x^*$$

Clearly $T^* \in \mathcal{L}(\mathcal{H})$ and

(44.2) $$\|T\| = \|T^*\|$$

The operator $T^*$ is called the <u>adjoint</u> of $T$ and is related to $T'$ in the following way. We have

(44.3) $$(x, Ty) = (T^*x, y) \quad , \forall\, x, y \in \mathcal{H}$$

In terms of the (anti-linear) map $\psi$ on p.27 taking $\mathcal{H} \to \mathcal{H}' \cong \mathcal{H}$, (44.3) takes the form $\langle \psi(T^*x), y \rangle = (T^*x, y)$ $= (x, Ty) = \langle \psi(x), T(y) \rangle = \langle T'\psi(x), y \rangle$ and so

$$\psi \circ T^* = T' \circ \psi$$

or

(44.4) $$T^* = \psi^{-1} \circ T' \circ \psi$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is <u>self-adjoint</u> if $T = T^*$.

The following important result of Banach provides information on the geometry of $\operatorname{ran} T$ for any bounded linear map $T$ from a Banach space $X$ to a Banach space $Y$. The defining relation

$$\langle T'y', x\rangle = \langle y', Tx\rangle \quad , x \in X, y' \in Y'$$

shows that

(45.1) $\ker T' = \{ y' : \langle y', y\rangle = 0 \quad \forall y \in \operatorname{ran} T\}$

On the other hand

(45.2) $\operatorname{ran} T \subset \{ y : \langle y', y\rangle = 0 \quad \forall y' \in N(T')\}$

but in general the inclusion is strict. Indeed the set on the right hand side of (45.2) is always closed, whereas ran $T$ may not be closed. However, as we now show, if ran $T$ is closed, then we have equality in (45.2)

<u>Theorem 45.3</u> (Banach) Let $X$ and $Y$ be Banach spaces and let $T \in \mathcal{L}(X, Y)$. Then the following properties are equivalent:

(45.4) ran $T$ is closed in $Y$

(45.5) ran $T'$ is closed in $X'$

(46.1) $\quad \operatorname{ran} T = \{ y \in Y : \langle y', y \rangle = 0 \ \ \forall\, y' \in \ker T' \}$

(46.2) $\quad \operatorname{ran} T' = \{ x' \in X' : \langle x', x \rangle = 0 \ \ \forall\, x \in \ker T \}$ $\square$

In order to prove the Theorem, it is convenient to use the following lemma

<u>Lemma 46.3</u> Let $X_i$, $1 \le i \le 4$, be Banach spaces. Suppose that

- $T_1 \in \mathcal{L}(X_1, X_2)$ is surjective
- $T_2 \in \mathcal{L}(X_2, X_3)$
- $T_3 \in \mathcal{L}(X_3, X_4)$ is injective with closed range
- $T_4 \in \mathcal{L}(X_1, X_4)$

Suppose that $T_4 = T_3 T_2 T_1$. Then

(46.4) $\quad \operatorname{ran} T_4$ is closed $\iff$ $\operatorname{ran} T_2$ is closed.

<u>Proof:</u>

$$\begin{tikzcd} X_1 \arrow[r, "T_1"] \arrow[rrr, bend right, "T_4"'] & X_2 \arrow[r, "T_2"] & X_3 \arrow[r, "T_3"] & X_4 \end{tikzcd}$$

Suppose $\operatorname{ran} T_2$ is closed and let $T_4 x_n \to y$, $x_n \in X_1$, $y \in X_4$. By (28.2), $\| T_3 T_2 T_1 x \| \ge c \| T_2 T_1 x \|$, $c > 0$, $\forall\, x \in X_1$, and so

$\{T_2 T_1 x_n\}$ is Cauchy : Suppose $T_2 T_1 x_n \to u \in X_3$. But as ran $T_2$ is closed, we must that $u = T_2 v$ for some $v \in X_2$. However, $T_1$ is surjective and hence $v = T_1 w$ for some $w \in X_1$. Then $y = \lim_{n\to\infty} T_4 x_n = \lim_{n\to\infty} T_3 (T_2 T_1 x_n) = T_3 u = T_3 T_2 v = T_3 T_2 T_1 w$ $= T_4 w$ and so ran $T_4$ is closed.

Conversely, suppose that ran $T_4$ is closed and let $T_2 v_n \to u$, $v_n \in X_2$, $u \in X_3$. As $T_1$ is surjective, this implies $T_2 T_1 w_n \to u$ for some $w_n \in X_1$, and hence $T_4 w_n \to T_3 u$. But as ran $T_4$ is closed, we must have $T_3 u = T_4 w = T_3 T_2 T_1 w$ for some $w \in X_1$, and hence $u = T_2 T_1 w$ as $T_3$ is injective. It follows that $u \in$ ran $T_2$ and so ran $T_2$ is closed. $\square$

<u>Proof of Theorem 46.1</u> We show first that to prove the equivalence of (45.4) and (45.5), it is enough to consider the case where

where $T$ is injective. ~~and ($\overline{\operatorname{ran} T}$ is dense in $Y$.)~~ Suppose $T \in \mathcal{L}(X, Y)$. Then

(48.1) $$T = I\,[T]\,\pi$$

where

- $\pi$ is the map $x \mapsto [x]$ taking $X$ onto $X/\ker T$
- $[T]$ is the map $[x] \mapsto Tx$ taking $X/\ker T$ into $\overline{\operatorname{ran}[T]} = \overline{\operatorname{ran} T}$, and
- $I$ is the map $y \to y$ injecting $\overline{\operatorname{ran} T}$ into $Y$

Clearly $\operatorname{ran}[T]$ is closed $\iff$ $\operatorname{ran} T$ is closed. Taking adjoints in (48.1) we obtain

(48.2) $$T' = \pi'[T]'I' \in \mathcal{L}(Y', X')$$

where
$$I' \in \mathcal{L}(Y', (\overline{\operatorname{ran} T})')$$
$$[T]' \in \mathcal{L}((\overline{\operatorname{ran} T})', (X/\ker T)')$$
$$\pi' \in \mathcal{L}((X/\ker T)', X')$$

If $y' \in Y'$ and $y \in \overline{\operatorname{ran} T}$, then $\langle I'y', y\rangle = \langle y', Iy\rangle = \langle y', y\rangle$

and we see that $I'y'$ is simply the restriction of $y'$ to $\overline{\operatorname{ran} T}$. But by the Hahn-Banach Theorem, any $y_0' \in (\overline{\operatorname{ran} T})'$ can be extended to an element $y' \in Y'$; clearly $I'y' = y_0'$. It follows that $I'$ is surjective. On the other hand, if $[x]' \in (X/\ker T)'$ and $x \in X$, then $\langle \pi'[x]', x \rangle = \langle [x]', \pi x \rangle = \langle [x]', [x] \rangle$. As $\pi$ is surjective, it follows immediately that $\pi'$ is injective. We show that $\operatorname{ran} \pi'$ is closed. First note that if $x' = \pi'[x]'$, then clearly $x'(x) = \langle \pi'[x]', x \rangle = \langle [x]', \pi x \rangle = \langle [x]', 0 \rangle = 0$ for all $x \in \ker T$. Conversely suppose that $x'(x) = 0$ for all $x \in \ker T$. Define the linear functional on $X/\ker T$

$$l([x]) = x'(x)$$

This functional is well defined: if $[x] = [y]$ then $x - y \in \ker T$ and hence $x'(x) = x'(y)$. Also $|l([x])| \le \|x'\| \, \|\tilde{x}\|$ for any $\tilde{x}$ in $[x]$, and as we can always choose $\tilde{x}$ such that $\|\tilde{x}\| \le 2 \|[x]\|$, we see that $l$ is bounded and hence $l \in (X/\ker T)'$. But $\langle \pi' l, x \rangle$

$= \langle \ell, \pi x \rangle = \langle \ell, [x] \rangle = x'(x) = \langle x', x \rangle$, which implies that

$\pi' \ell = x'$. Thus we have shown that

(50.1) $\quad \operatorname{ran} \pi' = \{ x' \in X' : x'(x) = 0 \ \forall x \in \ker T \}$

and it follows, in particular, that $\operatorname{ran} \pi'$ is closed. Applying Lemma 46.1, we conclude that

$$\operatorname{ran} T' \text{ is closed} \iff \operatorname{ran} [T]' \text{ is closed in } (X/\ker T)'$$

The above calculations show that to prove the equivalence of (45.4) and (45.5) it is enough to show that

(50.2) $\quad \operatorname{ran}[T] \text{ is closed} \iff \operatorname{ran}[T]' \text{ is closed}$

As $[T]$ is injective, we see that it is enough to prove the equivalence of (45.4) and (45.5) when $T$ is injective.

So suppose that $T \in \mathcal{L}(X, Y)$ is injective with closed range $\operatorname{ran} T = Y_1 \subset Y$. Now regard $T$ as a bijection $T_1$ from $X$ onto the Banach space $Y_1$. It follows from Theorem 41.2 that $T_1'$ is a bijection from $Y_1'$ onto $X'$. In particular for any $x' \in X'$, there exists $y_1' \in Y_1'$ such that $T_1' y_1' = x'$. Thus for any $x \in X$,

$$\langle T_1' y_1', x \rangle = \langle y_1', T_1 x \rangle = \langle y_1', Tx \rangle = \langle x', x \rangle$$

As $Y_1 \subset Y$, we may extend $y_1'$ to a bounded linear functional $y'$ on $Y$. Then $T'y' \in X'$ and we have for any $x \in X$,

$$\langle T'y', x \rangle = \langle y', Tx \rangle = \langle y_1', Tx \rangle = \langle x', x \rangle$$

so $T'y' = x'$. Thus $\operatorname{ran} T' = X$ so that, in particular, $\operatorname{ran} T'$ is closed.

Conversely, suppose that (T is one-to-one and) $\operatorname{ran} T'$ is closed in $X'$. We must show that $\operatorname{ran} T$ is closed. Let $Y_1 = \operatorname{ran} T \subset Y$ and let $T_1 : X \to Y_1$ as before. We show ~~first~~ that for $T_1' : Y_1' \to X'$

$$\operatorname{ran} T' \text{ closed} \Rightarrow \operatorname{ran} T_1' \text{ closed}.$$

Let $T_1' y_{1,n} \to x'$, $y_{1n} \in Y_1'$, $x' \in X'$

For each $n$, let $y'_n \in Y'$ be an extension to $y'_{1n}$ to $Y$. Then $\langle T' y'_n, x \rangle = \langle y'_n, Tx \rangle$

$= \langle y'_{1,n}, Tx \rangle$ and $Tx \in Y_1$

$= \langle y'_{1,n}, T_1 x \rangle = \langle T'_1 y'_{1n}, x \rangle$. Thus

$$T' y'_n = T'_1 y'_{1n} \to x'$$

and as ran $T'$ is closed, $\exists\ y' \in Y'$ st $T'y' = x'$ i.e.

$\langle x', x \rangle = \langle T'y', x \rangle = \langle y', Tx \rangle = \langle y'_1, Tx \rangle$, where

$y'_1$ is the restriction of $y'$ to $Y_1$

$= \langle y'_1, T_1 x \rangle = \langle T'_1 y'_1, x \rangle$ and so $x' = T'_1 y'_1$.

This shows that ran $T'_1$ is closed.

Following [Yosida], we now show the following fact: for any $\varepsilon > 0$, $\exists\ n = n(\varepsilon)$ st

(52.1) $$\{ y \in Y_1 : \|y\| \le \tfrac{1}{n} \} \subset \overline{\{ T_1 x : \|x\| < \varepsilon \}} \equiv S_\varepsilon$$

If not, $\exists\ y_n \in Y_1, y_n \to 0$, $y_n \notin S_\varepsilon$. But $S_\varepsilon$ is clearly a closed, convex balanced set and hence by Th$^m$ 27.1, $\exists\ y_n' \in Y_1'$ st

(53.1) $\quad y_n'(y_n) > 1 \geq |y_n'(T_1 x)| = |T_1' y_n'(x)|$ for all $\|x\| < \varepsilon$.

It follows that $\|T_1' y_n'\| \leq \frac{1}{\varepsilon} \|y_n'\|\, \|y_n\|$. On the other hand, as ran $T_1$ is dense in $Y_1$, $T_1'$ is injective and so as ran $T_1'$ is closed, $\|T_1' y'\| \geq c \|y'\|$ for some $c > 0$, $\forall\ y' \in Y_1'$, by (28.2). Thus

$$\frac{1}{\varepsilon} \|y_n'\|\, \|y_n\| \geq \|T_1' y_n'\| \geq c \|y_n'\|$$

$$\Rightarrow \qquad \|y_n\| \geq \varepsilon c$$

which contradicts $y_n \to 0$. This proves (52.1).

Next we show that for any $\varepsilon > 0$ we must have

(53.2) $\quad \{y \in Y_1 : \|y\| \leq \frac{1}{n}\} \subset \{Tx : \|x\| \leq \varepsilon\}$

for sufficiently large $n$. This then proves that

$\operatorname{ran} T_1 = Y_1$, for if $0 \neq y \in Y_1$, then $\| \frac{y}{n\|y\|} \| = \frac{1}{n}$

and so $\frac{y}{n\|y\|} = T_1 x$ for some $\|x\| \leq \varepsilon$. Hence

$y = T_1(n\|y\| x) \in \operatorname{ran} T_1$. As $\operatorname{ran} T_1 \subset \operatorname{ran} T$,

and so $\operatorname{ran} T = Y_1$, which is closed by construction.

To prove (53.2), let $\varepsilon > 0$ be given and set

$\varepsilon_i = \varepsilon / 2^{i+1}$, $i \geq 0$. By (52.1) $\exists$ a sequence of positive

numbers $\eta_i \to 0$ st for $i \geq 0$,

$$B_i = \{ y \in Y_1 : \|y\| \leq \eta_i \} \subset \overline{\{T_1 x : \|x\| \leq \varepsilon_i\}}$$

Let $y \in B_0$. Then $\| y - T_1 x_0 \| < \eta_1$ for some

$\|x_0\| < \varepsilon_0$. As $y - T x_0 \in B_1$, there then $\exists$ $\|x_1\| \leq \varepsilon_1$

st $\| y - T x_0 - T x_1 \| \leq \eta_2$. Repeating this process,

we find a sequence $\{x_i\}$ with $\|x_i\| \leq \varepsilon_i$ st

$$\| y - T_1 (\sum_{i=0}^{n} x_i) \| \leq \eta_{n+1}, \qquad n \geq 0$$

But for $n > m$, $\| \sum_{i=m}^{n} x_i \| \leq \sum_{i=m}^{n} \varepsilon_i = \frac{1}{2} \sum_{i=m}^{n} \frac{1}{2^i}$

and so $\{\sum_{i=0}^{n} x_i\}$ is a Cauchy sequence. Hence

$\sum_{i=0}^{n} x_i \to x \in X$ and $\|y - T_1 x\| = 0$ i.e. $y = T_1 x$

Moreover $\|x\| \le \sum_{i=0}^{\infty} \varepsilon_i = \varepsilon$. This shows that

$$\{y \in Y_1 : \|y\| \le m_0\} \subset \{T_1 x : \|x\| \le \varepsilon\}$$

as desired. This completes the proof that (45.4) is equivalent to (45.5).

The implications (46.1) $\Rightarrow$ (45.4) and (46.2) $\Rightarrow$ (45.5) are clear as the RHS's are closed sets.

The implication (45.4) $\Rightarrow$ (46.1) is straight forward.

~~(46.2) are clear as The RHS's in (46.1) and (46.2) are closed sets.~~ Indeed, by (45.2), ran $T$ is a subset of the RHS of (46.1). Suppose $\exists\, y \in Y$ st $\langle y', y\rangle = 0$ $\forall\, y' \in \ker T'$, but $0 \ne y \notin \operatorname{ran} T$.

Then as ran $T$ is closed, $\exists$ by Prop. 26.1 $y' \in Y'$ $y'(y) \ne 0$ an $y'(Tx) = 0$ $\forall\, x \in X$. Thus $\langle T'y', x\rangle = \langle y', Tx\rangle = 0$, and we see that $y' \in \ker T'$

by $\langle y', y \rangle \neq 0$, which is a contradiction.

• It remains to prove that (45.5) $\Rightarrow$ (46.2).

As $\operatorname{ran} T' \subset$ RHS, we again only need to show that if $\langle x', x \rangle = 0 \;\; \forall x \in \ker T$, then $x' \in \operatorname{ran} T'$. For such an $x'$, define the linear functional $l$ on $\operatorname{ran} T$ by

$$l(y) \equiv x'(x)$$

for any $x$ st $y = Tx$. The functional is well-defined for if $Tx_1 = Tx_2$, then $x_1 - x_2 \in \ker T$ and hence $x'(x_1 - x_2) = 0$. As (45.5) $\Leftrightarrow$ (45.4) we know that $\operatorname{ran} T$ is closed and hence by (29.7) we can always choose $x \in X$ st $y = Tx$ and $c\|x\| \leq \|y\|$ for some $0 < c < \infty$, independent of $x$ and $y$. Thus

$$|l(y)| \leq \|x'\| \|x\| \leq \frac{1}{c} \|x'\| \|y\|$$

and so $l$ is a bound linear functional on $\operatorname{ran} T$

$= \overline{\operatorname{ran} T}$, Let $y'$ be an extension of $l$ to $Y$.

Then for all $x \in X$, $\langle T'y', x\rangle = \langle y', Tx\rangle$

$= l(Tx) = \langle x', x\rangle$, and so $T'y' = x'$ i.e. $x' \in \operatorname{ran} T'$.

This completes the proof of Banach's theorem (45.3).

Lecture 5

Remark 57.1 Suppose $T \in \mathcal{L}(X, Y)$, and hence $T' \in \mathcal{L}(Y', X')$, has closed range. It is interesting to apply (46.1) to $T'$. We have

(57.2) $\operatorname{ran} T' = \{x' \in X' : \langle x'', x'\rangle = 0 \ \forall\, x'' \in \ker T''\}$

But (46.2) also applies, so we also have

(57.3) $\operatorname{ran} T' = \{x' \in X' : \langle x', x\rangle = 0 \ \forall\, x \in \ker T\}$

Using the injection $\varphi : X \to X''$ sending $x \mapsto x'' = \varphi(x)$

$x''(x') = x'(x)$ (see (26.3)), we see that

$$\langle x'', x'\rangle = 0$$

for all $x'' \in \ker T''$, if and only if

$$\langle x'', x'\rangle = 0$$

for all $x'' \in M \equiv \ker T'' \cap \operatorname{ran} \varphi$. But in general $M \subsetneq \ker T''$,