dim ran $T < \infty$. Such operators can be represented in the form (exercise)

(68.1) $$Tx = \sum_{i=1}^{n} x_i'(x)\, y_i \quad , x \in X$$

for some independent set of vector $y_i \in Y$, $i = 1, \ldots, n$, $n = \dim(\operatorname{ran} T)$, and some independent set of bounded linear functionals $x_i' \in X'$, $i = 1, \ldots, n$. Finite rank operators are clearly compact (why?). In Hilbert space, every compact operator $T \in K(H)$ is the norm limit of finite rank operators, $\|T_n - T\| \to 0$ as $n \to \infty$, T_n finite rank. (see Reed-Simon, e.g.): This is not true in general in Banach space.

Lecture 6

Theorem 68.2 (Schauder)

An operator $T \in \mathcal{L}(X, Y)$ is compact if and only if $T' \in \mathcal{L}(Y', X')$ is compact.

Proof: Recall the Arzela-Ascoli Theorem (see [Yosida]): Let S be a compact metric space and let $C(S)$ denote the B-space of continuous functions on S with norm $\|x\| = \sup_{s \in S} |x(s)|$.

Then a sequence of functions $\{x_n(s)\} \subset C(S)$ is relatively compact in $C(S)$ if the following two conditions are satisfied:

(1) $\{x_n\}$ is equibounded, i.e., $\sup_{n \geq 1} \|x_n\| < \infty$

(2) $\{x_n\}$ is equicontinuous, i.e.,

$$\lim_{\delta \downarrow 0} \sup_{n \geq 1,\, dist(s,s') < \delta} |x_n(s) - x_n(s')| = 0.$$

Now suppose that T is compact. Let $B = \{\|x\| < 1\}$ denote the unit ball in X. Then $S = \overline{T(B)}$ is a compact metric space. Now suppose that $\{y_j'\}$ is any sequence in B', the unit ball in Y', and consider the functions $f_i(y) = \langle y_i', y \rangle$ on $S \subset Y$. Then as

$$|f_i(y) - f_i(z)| = |\langle y_i', y - z \rangle| \leq \|y - z\|$$

it follows that the f_i's are equicontinuous on S. Also

$$|f_i(y)| = |\langle y_i', y\rangle| \le \|y\| \le \|T\|$$

we see that the f_i's are equibounded on S. Hence by the Arzela-Ascoli theorem, a subsequence $\{f_{i_k}\}$ converges uniformly on S. In particular

$$\langle T' y_{i_k}', x\rangle = \langle y_{i_k}', Tx\rangle = f_{i_k}(Tx)$$

converges uniformly for $x \in B$, which implies that

$$\|T' y_{i_k}' - T' y_{i_\ell}'\| = \sup_{\|x\| \le 1} |\langle T' y_{i_k}' - T' y_{i_\ell}', x\rangle|$$

converges to 0 as $k, \ell \to \infty$. Thus $\{T' y_{i_n}'\}$ is Cauchy and hence T' is a compact operator.

Conversely, suppose that T' is compact. Then by the previous argument T'' is compact. Let φ be the isometric embedding of X in X'', $\varphi(x)(x') = x'(x)$, $\|\varphi(x)\| = \|x\|$. Then

$$\langle y', Tx\rangle = \langle T'y', x\rangle = \langle \varphi(x), T'y'\rangle = \langle T''\varphi(x), y'\rangle$$

and so

$$(71.1) \qquad \|Tx\| = \sup_{\|y'\| \le 1} |\langle y', Tx \rangle| = \sup_{\|y'\| \le 1} |\langle T''\varphi(x), y' \rangle| = \|T''\varphi(x)\|$$

Now if $\{x_n\} \subset B$, then $\varphi(x_n) \in B''$, the unit ball in X'', and hence some subsequence $\{T''\varphi(x_{n_m})\}$ is Cauchy, as T'' is compact. But then by (71.1)

$$\|Tx_{n_k} - Tx_{n_\ell}\| = \|T''\varphi(x_{n_k}) - T''\varphi(x_{n_\ell})\|$$

and so $\{Tx_n\}$ is Cauchy. Thus T is compact. □.

The goal of Riesz - Schauder Theory is to describe the spectrum of compact operators $V \in \mathcal{L}(X)$. The following result of Riesz plays a fundamental role in the theory.

Lemma 71.2

Suppose X is a B-space and $V \in K(X)$. Then if $\lambda \neq 0$, $\operatorname{ran}(\lambda - V)$ is closed.

<u>Proof</u>: Let $T = \lambda - V$ and let $[T] : X/\ker(T) \to X$, $[T][x] = Tx$. Clearly $\operatorname{ran} T = \operatorname{ran}[T]$ and so it is sufficient to show that $\operatorname{ran}[T]$ is closed. As $[T]$ is injective, this amounts to showing that for some $c > 0$

$$(72.1) \qquad \| [T][x] \| \geq c \, \| [x] \| \quad , \quad [x] \in X/\ker(T).$$

Suppose (72.1) fails. Then $\exists$ $[x_n]$, y_n with $y_n = [T][x_n] = (\lambda - V)x_n$, and $\|[x_n]\| = 1$, st $y_n \to 0$. Now we may always choose x_n st $\|x_n\| \leq 2$. By the compactness of V, we see that $x_n = \frac{1}{\lambda}(y_n + Vx_n)$ has a convergent subsequence, say $x_{n_k} \to \hat{x}$. But the map $x \mapsto [x]$ is continuous, and so $[x_{n_k}] \to [\hat{x}]$ and as $\|[x_{n_k}]\| = 1$, we have $\|[\hat{x}]\| = 1$. Taking limits, we see that

$$[T][\hat{x}] = \lim_{k \to \infty} [T][x_{n_k}] = \lim_{k \to \infty} y_{n_k} = 0$$

which contradicts the injectivity of $[T]$. It follows that (72.1) holds true and hence $\operatorname{ran} T$ is closed. $\square$

By Theorem 68.2, as $V \in K(X)$, $V' \in K(X')$ and hence by Lemma 71.2, $\operatorname{ran}(\lambda - V)$ and $\operatorname{ran}(\lambda - V')$ are closed. Applying Th^m 45.3, we then obtain the following result:

<u>Theorem 73.1</u> Suppose $T = \lambda - V$ where $V \in K(X)$ & $\lambda \neq 0$. Then

(73.2) $\operatorname{ran} T = \{ y \in X : \langle y', y \rangle = 0 \ \ \forall y' \in \ker T' \}$

and

(73.3) $\operatorname{ran} T' = \{ x' \in X' : \langle x', x \rangle = 0 \ \ \forall x \in \ker T \}$

We are now able to prove the main results of Riesz-Schauder Theory describing the spectrum of $V \in K(X)$. Part (1) of Theorem 73.4 below is known as the Fredholm Alternative. We follow [Yosida].

<u>Theorem 73.4</u> Suppose $V \in K(X)$. Then

(1) if $\lambda \neq 0$ is not an eigenvalue of V, then λ is in the resolvent set of V (Note: this property is well known for matrices!)

(2) $\operatorname{Spec} V \setminus \{0\}$ is a discrete set in $\mathbb{C}$, with $\lambda = 0$ as the only possible accumulation point

(3) if $0 \neq \lambda \in \operatorname{spec} V$, then $1 \le \dim(\ker(\lambda - V)) < \infty$

(4) if $\lambda \neq 0$, then $\dim \ker(\lambda - V) = \dim \ker(\lambda - V')$

(Note: this fact corresponds to the fact that "row rank = column rank" in the matrix case.)

<u>Proof</u>: Suppose $\lambda \neq 0$ is not an eigenvalue of V and set $T = \lambda - V$. Then T is injective and, by the above Lemma 71.2, $X_1 = \operatorname{ran} T$ is closed in X. Then for some

(74.1) $0 < c < \infty$, $c\|x\| \le \|Tx\| \quad \forall x \in X$. If $X_1 \subsetneq X_0 = X$,

it follows that $X_k \equiv T^k X$, $k \ge 0$, form a <u>strictly</u> decreasing chain of closed subspaces

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_k \supsetneq X_{k+1} \cdots$$

Indeed, as $TX \subset X$, we have $TX_k = T^{k+1}X = T^k(TX)$ $\subset T^k X = X_k$ so that T maps X_k into itself; $X_{k+1} = TX_k$ $\subset X_k$, and by (74.1), $c\|T^k x\| \le \|T^{k+1}x\| \quad \forall x$, and so X_{k+1} is closed in X_k. Finally if $X_{k+1} = X_k$, then for any $x \in X$, $T^k x \in X_{k+1}$ and so $T^k x = T^{k+1}x'$ for some $x' \in X$. But

Then $x = Tx'$ as T is 1-1, which contradicts $X_1 \subsetneq X_0$.

By Riesz's Theorem 14.1, $\exists\, x_k \in X_k$, $\|x_k\| = 1$ st

$\text{dist}(x_k, X_{k+1}) \geq \frac{1}{2}$. But if $k > \ell$

$$\frac{1}{\lambda}(Vx_\ell - Vx_k) = x_\ell + \left(-x_k - \frac{Tx_\ell}{\lambda} + \frac{Tx_k}{\lambda}\right) = x_\ell - x$$

for some $x \in X_{\ell+1}$. Hence $\|Vx_\ell - Vx_k\| \geq \frac{|\lambda|}{2}$, which contradicts the compactness of V. This proves (1).

Now it is a simple exercise to show that eigenvectors corresponding to distinct eigenvalues are independent, i.e., if $\sum_{i=1}^n \alpha_i u_i = 0$ and $(V - \lambda_i)u_i = 0$ with $u_i \neq 0$ and $\lambda_k \neq \lambda_j$ for $k \neq j$, then $\alpha_1 = \dots = \alpha_n = 0$. It follows that to prove (2) and (3) we must show that the following statement leads to a contradiction:

There exists a sequence $\{x_i\}$ of linearly independent vectors such that $Vx_i = \lambda_i x_i$, $i \geq 1$ and $\lim_{i \to \infty} \lambda_i = \lambda \neq 0$.

For such $\{x_i\}$, let X_n be the (closed) subspace spanned by $x_1, \dots, x_n$. Then again by Riesz's Theorem, $\exists\, y_n \in X_n$, $\|y_n\| = 1$ such that $\operatorname{dist}(y_n, X_{n-1}) \geq \frac{1}{2}$, $n \geq 2$. Set $T_n = \lambda_n - V$. Then for $n > m$

$$\frac{1}{\lambda_n} V y_n - \frac{1}{\lambda_m} V y_m = y_n + \left(-y_m - \frac{1}{\lambda_n} T_n y_n + \frac{1}{\lambda_m} T_m y_m\right).$$

But $z_{nm} \equiv y_m + \frac{1}{\lambda_n} T_n y_n - \frac{1}{\lambda_m} T_m y_m \in X_{n-1}$. For if $y_n = \sum_{j=1}^{n} \beta_j x_j$, then $T_n y_n = \sum_{j=1}^{n} \beta_j (\lambda_n - \lambda_j) x_j = \sum_{j=1}^{n-1} \beta_j (\lambda_n - \lambda_j) x_j \in X_{n-1}$ and similarly $T_m y_m \in X_m \subset X_{n-1}$. Hence

$$\left\| \frac{1}{\lambda_n} V y_n - \frac{1}{\lambda_m} V y_m \right\| = \| y_n - z_{nm} \| \geq \frac{1}{2}.$$

Combined with the fact that $\lambda_n \to \lambda \neq 0$ this contradicts the compactness of V. This proves (2) and (3)

Finally we note that as V is compact, V' is compact. Let $\lambda \neq 0$ and set $T = \lambda - V$. Part (1) together with Theorem 64.1 then implies that

$$\lambda \text{ is an eigenvalue of } T$$
$$\Longleftrightarrow$$
$$\lambda \in \operatorname{spec} T$$
$$\Longleftrightarrow$$
$$\lambda \in \operatorname{spec} T'$$
$$\Longleftrightarrow$$
$$\lambda \text{ is an eigenvalue of } T'$$

- ie $\lambda \neq 0$ is an eigenvalue of T if and only if λ is an eigenvalue of T'. The relation in (4) quantifies this statement and reflects the familiar fact for finite, square matrices "row rank = column rank". And in fact, as we will see, the proof of (4) reduces to the matrix case.

We prove (4) in 3 steps:

<u>Step 1</u> Suppose $0 \neq \lambda \in \sigma(T)$. Then by (2), λ is isolated, and for some $\varepsilon > 0$, we can expand $(s-V)^{-1}$ in a Laurent series

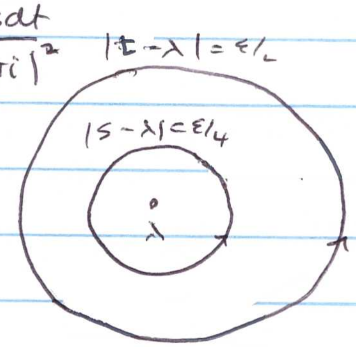
$$(s-V)^{-1} = \sum_{n=-\infty}^{\infty} (s-\lambda)^n A_n \tag{77.1}$$

for any $s \in \{t : 0 < |t-\lambda| < \varepsilon\} \subset \mathbb{C} \setminus \{0\}$. The residue term is given by the formula

$$A_{-1} = \int_{C_\varepsilon} (s-V)^{-1} \frac{ds}{2\pi i}$$

where C_ε is any simple anti-clockwise contour in $\mathbb{C}\setminus\{0\}$ enclosing λ and lying in $\{0 < |s-\lambda| < \varepsilon\}$. The operator A_{-1} is a (bounded) projection. Indeed A_{-1} is clearly bded (why?) and

$$A_{-1}^2 = \oint_{|s-\lambda| = \frac{\varepsilon}{4}} \oint_{|t-\lambda| = \frac{\varepsilon}{2}} (s-V)^{-1}(t-V)^{-1} \frac{ds\,dt}{(2\pi i)^2}$$



$$= \oint_{|s-\lambda| = \frac{\varepsilon}{4}} \oint_{|t-\lambda| = \frac{\varepsilon}{2}} \left(\frac{1}{s-V} - \frac{1}{t-V}\right) \frac{ds\,dt}{(t-s)(2\pi i)^2}$$

$$= \frac{1}{(2\pi i)^2} \oint_{|s-\lambda| = \frac{\varepsilon}{4}} \frac{ds}{s-V} \left(\oint_{|t-\lambda| = \frac{\varepsilon}{2}} \frac{1}{t-s}\,dt\right)$$

$$- \frac{1}{(2\pi i)^2} \oint_{|t-\lambda| = \frac{\varepsilon}{2}} \frac{dt}{t-V} \oint_{|s-\lambda| = \frac{\varepsilon}{4}} \frac{1}{t-s}$$

$$= \frac{1}{2\pi i} \int_{|s-\lambda| = \frac{\varepsilon}{4}} \frac{1}{s-V} \cdot 1\, ds \quad - \quad 0$$

$$= A_{-1}.$$

Also A_{-1} is compact, as we see from the calculation

$$A_{-1} = \int_{C_\varepsilon} \frac{1}{s-V} \frac{ds}{2\pi i} = \int_{C_\varepsilon} \left(\frac{1}{s} + \frac{1}{s}\frac{1}{s-V}V\right) \frac{ds}{2\pi i}$$

$$= \left(\int_{C_\varepsilon} \frac{1}{s} \frac{1}{s-V} \frac{ds}{2\pi i} \right) V \in K(X).$$

Now the range of any (bded) projection is closed and so ran A_{-1} is a closed subspace. Let S be the unit ball in ran A_{-1}. Now as $A_{-1}^2 = A_{-1}$ we must have $S = A_{-1} S$, and so S is compact. By Corollary 15.1, it follows that ran A_{-1} is finite dimensional.

Step 2

As V clearly commutes with A_{-1}, it follows that $TA_{-1} = A_{-1}T$ and hence T maps A_{-1} into itself. We claim that

(79.1) $$\ker T = \ker (T \upharpoonright \operatorname{ran} A_{-1})$$

Clearly RHS $\subset$ LHS. So suppose $x \in X$ and $Tx = 0$, i.e., $Vx = \lambda x$, and so $(s-V)^{-1}x = (s-\lambda)^{-1}x$ $\forall s \in \rho(V)$. We have

$$A_{-1}x = \int_{C_\varepsilon} \frac{1}{s-V} x \frac{ds}{2\pi i} = \int_{C_\varepsilon} \frac{1}{s-\lambda} x \frac{ds}{2\pi i} = x$$

Thus $x \in$ ran A_{-1}, which proves the claim.

Now $\sigma(V) = \sigma(V')$ and so λ must be an isolated pt in $\sigma(V')$. The same argument as above shows that T' maps ran A'_{-1} into itself and

(80.1) $$\ker T' = \ker (T' \restriction \operatorname{ran} A'_{-1})$$

Note that
$$(s - V)^{-1} = \sum_{-\infty}^{\infty} (s-\lambda)^n A_n$$

$\Rightarrow$
$$(s - V')^{-1} = \sum_{-\infty}^{\infty} (s-\lambda)^n A'_n$$

and so $(A_{-1})'$ is the residue term in the Laurent series for $(s - V')^{-1}$.

Let $(\operatorname{ran} A_{-1})'$ denote the (abstract) dual space of the finite dimensional space ran A_{-1}. Define the linear map

(80.2) $$h : \operatorname{ran}(A'_{-1}) \to (\operatorname{ran} A_{-1})'$$

as follows. For $x' \in \operatorname{ran}(A'_{-1}) \subset X'$, set

(80.3) $$h(x') = x' \restriction (\operatorname{ran} A_{-1})$$

ie if $x \in \operatorname{ran} A_{-1}$,

$$h(x')(x) = x'(x)$$

We now claim the following. The map h is a bijection from $\operatorname{ran} A'_{-1}$ onto $(\operatorname{ran} A_{-1})'$ and if $\tilde{T} : (\operatorname{ran} A_{-1})' \to (\operatorname{ran} A_{-1})'$ denotes the dual of $T \restriction \operatorname{ran} A_{-1}$, then on $\operatorname{ran} A'_{-1}$

(81.1) $$\tilde{T} \circ h = h \circ T'$$

Indeed, suppose $x' \in \operatorname{ran} A'_{-1}$ and $h(x') = 0$. Then $x'(x) = 0 \ \forall$ $x \in \operatorname{ran} A_{-1}$. Hence $A'_{-1} x'(x) = x'(A_{-1} x) = 0 \quad \forall x \in X$. But then as $x' \in \operatorname{ran} A'_{-1}$, we have $x' = A'_{-1} x' = 0$ and so h is 1-1. Now suppose $\tilde{l} \in (\operatorname{ran} A_{-1})'$, then

(81.2) $$x'(x) = \tilde{l}(A_{-1} x) \quad , \quad x \in X$$

defines a (bounded) linear functional $x' \in X'$. For all $x \in X$, we have $A'_{-1} x'(x) = x'(A_{-1} x) = \tilde{l}(A_{-1}^2 x) = l(A_{-1} x) = x'(x)$

and so $x' = A_{-1}' x' \in \operatorname{ran} A_{-1}'$, and for any $x \in \operatorname{ran} A_{-1}$

$$h(x')(x) = x'(x) = \tilde{\ell}(A_{-1} x) = \tilde{\ell}(x)$$

and so $\tilde{\ell} \in \operatorname{ran} h$. It follows that h is a bijection.

Finally, suppose $x' \in \operatorname{ran} A_{-1}'$ and $x \in \operatorname{ran} A_{-1}$. Then

$$\begin{aligned}(\tilde{T} h(x'))(x) &= h(x')(Tx) = x'(Tx) \\ &= T'x'(x) \\ &= h(T'x')(x),\end{aligned}$$

which proves (81.1).

<u>Step 3</u> It follows from (81.1) that $T' \restriction \operatorname{ran} A_{-1}'$ is similar to $\tilde{T}$, and so

(82.1) $$\dim \ker (T' \restriction \operatorname{ran} A_{-1}') = \dim \ker \tilde{T}$$

But $T \restriction \operatorname{ran} A_{-1}$ is similar to a finite square matrix M and $\tilde{T}$ is similar to its transpose M^T. By basic linear algebra (row rank = column rank), we see

(82.2) $$\dim \ker (T' \restriction \operatorname{ran} A_{-1}') = \dim \ker \tilde{T} = \dim \ker (T \restriction \operatorname{ran} A_{-1})$$

But then by (79.1) and (80.1), $\dim \ker T = \dim \ker T'$, as desired. This completes the proof of the Theorem. $\square$

Lecture 7

Remark 83.1

It follows, in particular, from the fact that h in (80.2) is a bijection, that for $\lambda \neq 0$

(83.2) $$\dim(\operatorname{ran} A'_{-1}) = \dim(\operatorname{ran} A_{-1})$$

The common value in (83.2) is called the algebraic multiplicity of λ for V, and also for V'.

Exercise 83.3 Show that the algebraic multiplicity is $\geq$ geometric multiplicity. Also show that in the matrix case, the algebraic mult. in (83.2) coincides with the matrix defn. of alg. multiplicity.