Lecture 7 The following Theorem describes the basic relation between the co-dimension of T (resp. T') and the dimension of the null space of T' (resp. T).

Theorem 84.1 Let $T \in \mathcal{L}(X, Y)$. Suppose $\operatorname{ran} T$, or equivalently, $\operatorname{ran} T'$, is closed. Then

(84.2) $$\operatorname{codim} T = \dim \ker(T')$$

(84.3) $$\operatorname{codim} T' = \dim \ker(T)$$

Remark 84.4 We interpret (84.2) in the sense that $\operatorname{codim} T$ and $\dim \ker(T')$ are simultaneously finite or infinite, and if they are finite, then they are equal. The interpretation of (84.3) is similar.

Proof of Theorem 84.1: Consider first (84.2) and suppose that $\operatorname{codim} T = n < \infty$. The case $n = 0$ is clear, so assume that $0 < n < \infty$. Let $\{u_1, \dots, u_n\}$ be a basis for $Y/\operatorname{ran} T$. For each

$i \in \{1, \dots, n\}$, choose $y_i \in u_i$. Then each $y \in Y$ has a unique representation $y = \lambda_1 y_1 + \cdots + \lambda_n y_n + r$ where $\lambda_i \in \mathbb{C}$ and $r \in \operatorname{ran} T$.

Define $y_i'(y) = y_i'(\lambda_1 y_1 + \cdots + \lambda_n y_n + r) \equiv \lambda_i$, $1 \le i \le n$.

Now $\pi : Y \to Y/\operatorname{ran} T$ is bounded with $\|\pi\| = 1$ (see (22.2)).

Thus for some $c > 0$, $\|y\| \ge \|\pi y\| = \|\lambda_1 u_1 + \cdots + \lambda_n u_n\|$

$\ge c(|\lambda_1| + \cdots + |\lambda_n|) \ge c|\lambda_i| = c|y_i'(y)|$, $1 \le i \le n$. Thus each y_i' is bded. Observe that $y_i'(y_j) = \delta_{ij}$, $1 \le i, j \le n$.

By (45.1),

$\ker T' = \{y' \in Y' : \langle y', y \rangle = 0 \ \forall y \in \operatorname{ran} T\}$.

Hence $y_i' \in \ker T'$, $1 \le i \le n$.

Define the mapping $f : Y/\operatorname{ran} T \to \ker T'$ by

$f(\lambda_1 u_1 + \cdots + \lambda_n u_n) = \lambda_1 y_1' + \cdots + \lambda_n y_n'$. We show that f is a bijection. Indeed if $\lambda_1 y_1' + \cdots + \lambda_n y_n' = 0$, then evaluation at y_j shows that $\lambda_j = 0$. Hence f is 1-1.

On the other hand, if $y' \in \ker T'$, define

$$\tilde{y}' = y' - \sum_{i=1}^{n} y'(y_i)\, y_i'$$

Then $\tilde{y}'(y_i) = y'(y_i) - y'(y_i) = 0$, $1 \le i \le n$. Also $\tilde{y}'(r) = 0$ for $r \in \operatorname{ran} T$. Hence $\tilde{y}' = 0$ and $y' = \sum_{i=1}^{n} y'(y_i) g_i' \in \operatorname{ran} f$

This shows that if $\dim Y/\operatorname{ran} T < \infty$, then $\dim \ker T' < \infty$ and (84.2) holds. Now if $\{u_1, \dots, u_n\}$ is any set of independent vectors in $Y/\operatorname{ran} T$, the above construction yields n independent, bounded linear functionals y_i' on the subspace (with the property that $y_i' \restriction \operatorname{ran} T = 0$) $\{\lambda_1 y_1 + \dots + \lambda_n y_n + r : \lambda_i \in \mathbb{C},\ r \in \operatorname{ran} T\} \subset Y$. By the Hahn Banach Theorem these functionals can be extended to (independent) bounded linear functionals on Y. If Y_i' is the extension of y_i' then for $y \in Tx \in \operatorname{ran} T$,

$$\langle Y_i', y \rangle = \langle Y_i', Tx \rangle = \langle y_i', Tx \rangle = 0$$

so $Y_i' \in \ker T'$. Thus $\dim \ker T' \geq n$. It follows that if $Y/\operatorname{ran} T$ is not finite dimensional, then the same is true for $\ker T'$. This proves (84.2) in the sense of Remark 84.4

above.

Now consider (84.3) and suppose that $\dim \ker T = n < \infty$. Let $x_1, \dots, x_n$ be a basis for $\ker T$. By the Hahn-Banach Theorem there exist functionals $x_i' \in X'$, $1 \le i \le n$, such that $x_i'(x_j) = \delta_{ij}$, $1 \le i < j \le n$. Consider the map $g : \ker T \to X'/\operatorname{ran} T'$ taking $\lambda_1 x_1 + \dots + \lambda_n x_n \mapsto \lambda_1 [x_1'] + \dots + \lambda_n [x_n']$

We claim that g is a bijection. Indeed if $\lambda_1 [x_1'] + \dots + \lambda_n [x_n'] = 0$, then $\lambda_1 x_1' + \dots + \lambda_n x_n' + x' = 0$ for some $x' \in \operatorname{ran} T'$. Evaluating at x_k, we find $\lambda_k + x'(x_k) = 0$. But as $x' = T'y'$ for some $y' \in Y'$, $x'(x_k) = T'y'(x_k) = y'(Tx_k) = 0$. Thus $\lambda_k = 0$, $k = 1, \dots, n$, and so g is injective. On the other hand, suppose $x' \in X'$ and set $\tilde{x}' = x' - \sum_{i=1}^{n} x'(x_i) x_i'$. Then $\tilde{x}'(x_i) = 0$ and so $\langle \tilde{x}', x \rangle = 0$ for all $x \in \ker T$. But $\operatorname{ran} T'$ is closed, and hence by (46.2)

$$\operatorname{ran} T' = \{x' : \langle x', x\rangle = 0 \text{ for all } x \in \ker T\}.$$

Thus $\tilde{x}' \in \operatorname{ran} T'$ and it follows that $[x'] = \sum_{i=1}^{n} x'(x_i)[x_i']$

This shows that g is onto and hence $\operatorname{codim} T' = n = \dim \ker T$.

Finally, as in the proof of (84.2), we note that if $\{x_1, \dots, x_n\}$ is any set of independent vectors in $\ker T$, the above construction yields n independent vectors $\{[x_1'], \dots, [x_n']\}$ in $X'/\operatorname{ran} T'$. Thus $\operatorname{codim} T' \geq n$. It then follows that if $\ker T$ is not finite dimensional, the same is true for $X'/\operatorname{ran} T'$. This proves (84.3) in the sense of Remark 84.4 ☐

Let X and Y be Banach spaces, and suppose $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$. Then the relationship between the operators $AB \in \mathcal{L}(Y)$ and $BA \in \mathcal{L}(X)$, and in particular the relationship between $\sigma(AB)$ and $\sigma(BA)$, plays a central role in a surprisingly broad spectrum of problems in analysis, including Riemann–Hilbert Theory. Of course, if A, or similarly B, is a bijection, then $BA = A^{-1}(AB)A$ and AB are clearly similar. But much more is true. Direct computation yields the following <u>commutation formula</u>: if $\lambda \neq 0$, then

(89.1) $$\frac{\lambda}{AB+\lambda} + A\frac{1}{BA+\lambda}B = 1$$

in the sense that if $-\lambda \in \rho(BA)$, $-\lambda \in \rho(AB)$ and $\frac{1}{\lambda}\left(1 - A\frac{1}{BA+\lambda}B\right)$ provides an inverse for $AB+\lambda$, and vice-versa. In particular

(89.2) $$\sigma(AB)\setminus\{0\} = \sigma(BA)\setminus\{0\}$$

Formula (89.1) can be found, e.g. in S. Sakai, C^*-algebras and W^*-algebras, Springer, 1971.

An extension of the formula to unbounded operators A, B is given in P. Deift, Duke Math. J 45 (1978), 267–310.

together with many applications to scattering theory and inverse scattering theory. The formula may be viewed as providing a basic isospectral action on operators. The commutation formula (89.1) also has an infinitesimal version (see Deift, D.M. Journal above).

The following result extends further the relationship between AB and BA

Thm 90.1

Let X and Y be Banach spaces and let $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$. Suppose $\lambda \neq 0$. Then

(90.2) $\quad \operatorname{ran}(\lambda + AB)$ is closed $\iff$ $\operatorname{ran}(\lambda + BA)$ is closed

(90.3) $$\dim \ker(\lambda + AB) = \dim \ker(\lambda + BA)$$

(90.4) $$\operatorname{codim}(\lambda + AB) = \operatorname{codim}(\lambda + BA)$$

Proof: First consider (90.2). By symmetry it is clearly sufficient to show that if $\operatorname{ran}(\lambda + AB)$ is closed, then $\operatorname{ran}(\lambda + BA)$ is closed. So suppose $\operatorname{ran}(\lambda + AB)$ is closed and $(\lambda + BA)x_n \to \hat{x}$ for $x_n, \hat{x} \in X$. Then $(\lambda + AB)Ax_n \to A\hat{x}$ and so $(\lambda + AB)y = A\hat{x}$ for some $y \in Y$. But then $(\lambda + BA)By = B(\lambda + AB)y$ $= BA\hat{x} = (\lambda + BA)\hat{x} - \lambda\hat{x}$ and so $(\lambda + BA)x^{\#} = \hat{x}$ where $x^{\#} = \lambda^{-1}(\hat{x} - By)$. This shows that $\operatorname{ran}(\lambda + BA)$ is closed, proving (90.2)

To prove (90.3), we claim that the map

$$f : \ker(\lambda + BA) \to \ker(\lambda + AB)$$

taking $x \mapsto Ax$ is a bijection. Indeed, if $(\lambda + BA)x = 0$ then $(\lambda + AB)Ax = 0$, and if $Ax = 0$, then $x = -\frac{1}{\lambda}BAx = 0$. Thus f is injective. On the other hand, if $0 \neq y \in$ $\ker(\lambda + AB)$, then $\frac{1}{\lambda}By \in \ker(\lambda + BA)$ and $A(-\frac{1}{\lambda}By) =$

$-\frac{1}{\lambda}[(\lambda + AB)y] + y = y$. This shows that f is a bijection, proving (90.3).

Finally, to prove (90.4), we claim that

$$g : X/\operatorname{ran}(\lambda + BA) \to Y/\operatorname{ran}(\lambda + AB)$$

taking $[x] \mapsto [Ax]$ is a bijection. Indeed, if $x, \hat{x} \in X$, then $A(x + (\lambda + BA)\hat{x}) = Ax + (\lambda + AB)A\hat{x}$, and so g is well-defined. Moreover, if $[Ax] = 0$, then $Ax + (\lambda + AB)y = 0$ for some $y \in Y$, and so $x = \frac{1}{\lambda}(\lambda + BA)(x + By)$ $\in \operatorname{ran}(\lambda + BA)$, and so $[x] = 0$. Thus g is injective. Finally, suppose $[y] \in Y/\operatorname{ran}(\lambda + AB)$. Let $x = -\frac{1}{\lambda}By$ for any $y \in [y]$. Then $Ax = -\frac{1}{\lambda}ABy = y - \frac{1}{\lambda}(\lambda + AB)y$. Thus $g(x) = [Ax] = [y]$ and so g is onto. This proves (90.4) and completes the proof of the Theorem. $\square$

Combining Theorem 90.1 with Theorem 84.1, we obtain the following result.

Th^m 93.1 Let X and Y be Banach spaces and suppose $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$. Suppose $\lambda \neq 0$. Then

(93.2) $\operatorname{ran}(\lambda + AB)$ is closed if and only if $\operatorname{ran}(\lambda + BA)$ is closed if and only if $\operatorname{ran}(\lambda + B'A')$ is closed if and only if $\operatorname{ran}(\lambda + A'B')$ is closed.

Suppose that one, and hence all, of the operators in (93.2) have closed range. Then

(93.3) $$\operatorname{codim}(\lambda + AB) = \operatorname{codim}(\lambda + BA) = \dim\ker(\lambda + B'A') = \dim\ker(\lambda + A'B')$$

(93.4) $$\operatorname{codim}(\lambda + A'B') = \operatorname{codim}(\lambda + B'A') = \dim\ker(\lambda + BA) = \dim\ker(\lambda + AB)$$

Insert after Remark 84.4.

Remark 93.5 If $\operatorname{ran} T$ is not closed, relation (84.2) may fail. Indeed, if $\operatorname{ran} T$ is dense in Y, but $\operatorname{ran} T \neq Y$, then $\operatorname{codim} T > 0$, but $\dim \ker T' = 0$.

Let $V \in K(X)$ and set $T = 1 - V$. The <u>index</u> of T is defined by

(94.1) $$\operatorname{ind} T = \dim \ker T - \operatorname{codim} T$$

Note that $\dim \ker T$ and $\operatorname{codim} T = \dim \ker T'$ both finite by Schauder's Theorem (68.2), lemma 71.2, Theorem 73.4 (4) (84.3). For square matrices $\operatorname{ind} T = 0$ ("row rank = column rank"). The same is true for $T = 1 - V$, $V \in K(X)$, as we now show. This result is of fundamental importance.

<u>Thm 94.2</u> Let $V \in K(X)$ and set $T = 1 - V$. Then

(94.3) $$\operatorname{ind} T = 0 .$$

<u>Proof:</u> By Theorem 73.4 (4) and (84.2)

$$\dim \ker T = \dim \ker T' = \operatorname{codim} T$$

This proves (94.3). $\square$

More can be said in general about the structure of compact operators in the case that $X = \mathcal{H}$ is a Hilbert space. Note first that for a general B-space X we say that a sequence of vectors x_n <u>converges weakly</u> to $x \in X$, written $x_n \rightharpoonup x$, if

(95.1) $$x'(x_n) \to x'(x) \quad \text{for all } x' \in X'.$$

If $X = \mathcal{H}$, this means that

(95.2) $$(y, x_n) \to (y, x) \quad \forall y \in \mathcal{H} \cong \mathcal{H}',$$

Convergence $x_n \to x$ in the norm of X is called <u>strong convergence</u>. Clearly strong conv. $\Rightarrow$ weak conv.

The following important result is due to Eberlein and Shmulyan. The proof in the general reflexive case is given for example, in [Yosida]. The proof in the Hilbert space case is left as an <u>exercise</u>.

(Eberlein & Shmulyan)

Th^m 96.1 A Banach space X is reflexive if and only if every bounded sequence in X contains a subsequence which converges weakly to an element of X. □

In particular this result applies to Hilbert space, and also to $L^p(M, \mathcal{A}, \mu)$ for $1 < p < \infty$. Weak convergence provides a very useful characterization for operators in a reflexive space.

Th^m 96.2 Let X be a reflexive Banach space and let $V \in \mathcal{L}(X)$. Then V is compact $\Leftrightarrow$ V takes weakly convergent sequences to strongly convergent sequences.

Proof: Suppose V is compact and $x_n \rightharpoonup x$. Then by the uniform bdedness principle $\|x_n\| \le c < \infty$. Now for $x' \in X'$,

$x'(Vx_n) = V'x'(x_n) \to V'x'(x) = x'(Vx)$ and so $Vx_n \rightharpoonup Vx$. Suppose $Vx_n \not\to Vx$. Then for some $\varepsilon > 0$, $\exists$ a subseq.

Vx_{n_k} such that $\| Vx_{n_k} - Vx \| \geq \varepsilon$. But as $\|x_n\| \leq c$, $\{x_{n_k}\}$ has a further subsequen $\{x_{n_k'}\}$, say, such that $Vx_{n_k'} \to \tilde{y}$. Necessarily $\| \tilde{y} - Vx \| \geq \varepsilon$. But $Vx_{n_k'} \to \tilde{y}$ $\Rightarrow$ $Vx_{n_k'} \rightharpoonup \tilde{y}$ and so $\tilde{y} = Vx$. But this is impossible, and so we must have $Vx_n \to Vx$.

Conversely, suppose V takes weakly conv. subseq's to strongly conv. subsequences. Suppose that $\| x_n \| \leq c$. Then by Th^m 96.1, $\{x_n\}$ has a weakly convergent subsequence, $\{x_{n_k}\}$, $x_{n_k} \rightharpoonup x$ for some x, and then by the assumption, $Vx_n \to Vx$. This proves the Theorem. $\square$.

By Theorem 67.1 (3), the norm limit of finite rank operators is compact. In a separable Hilbert space, the converse is true.

Lecture 8

Th^m 97.1 Let $\mathcal{H}$ be a separable Hilbert space. Then every