Vx_{n_k} such that $\|Vx_{n_k} - Vx\| \geq \varepsilon$. But as $\|x_n\| \leq c$, $\{x_{n_k}\}$ has a further subsequence $\{x_{n_k'}\}$, say, such that $Vx_{n_k'} \to \tilde{y}$. Necessarily $\|\tilde{y} - Vx\| \geq \varepsilon$. But $Vx_{n_k'} \rightharpoonup \tilde{y}$ $\Rightarrow$ $Vx_{n_k'} \rightharpoonup \tilde{y}$ and so $\tilde{y} = Vx$. But this is impossible, and so we must have $Vx_n \to Vx$.

Conversely, suppose V takes weakly conv. subseq's to strongly conv. subsequences. Suppose that $\|x_n\| \leq c$. Then by Th^m 96.1, $\{x_n\}$ has a weakly convergent subsequence, $\{x_{n_k}\}$, $x_{n_k} \rightharpoonup x$ for some x, and then by the assumption, $Vx_n \to Vx$. This proves the Theorem. $\square$.

By Theorem 67.1 (3), the norm limit of finite rank operators is compact. In a separable Hilbert space, the converse is true.

Lecture 8

Th^m 97.1 Let H be a separable Hilbert space. Then every compact operator V is the norm limit of a sequence of finite rank operators (cf. discussion on p. 68 and also Remark (99.1) below.)

<u>Proof</u> (Following Reed–Simon I)

As $\mathcal{H}$ is separable, it has a countable orthonormal basis $\{x_n, n \geq 1\}$

Define

$$a_n = \sup_{x \in \langle x_1, \ldots, x_n \rangle^\perp,\ \|x\|=1} \|Vx\|$$

Clearly a_n is decreasing: $a_n \downarrow a$, $a \geq 0$. We show $a = 0$.

Suppose $a > 0$. Choose u_n, $u_n \in \langle x_1, \ldots, x_n \rangle^\perp$ with $\|u_n\| = 1$ such that $\|Vu_n\| \geq \frac{a}{2}$. Let $x \in \mathcal{H}$ and let $\varepsilon > 0$ be given. As $\{x_n\}$ is an orthonormal basis, we can write $x = \hat{x} + \tilde{\varepsilon}$ where $\hat{x} \in \langle x_1, \ldots, x_k \rangle$ for some $k \geq 1$ and $\|\tilde{\varepsilon}\| < \varepsilon$.

Then $(x, u_n) = (\hat{x}, u_n) + O(\varepsilon) = O(\varepsilon)$ for $n > k$. This shows that $u_n \rightharpoonup 0$ and hence $Vu_n \to 0$ by the compactness of V. This is a contradiction and it follows that $a = 0$.

For any $x \in H$, $\|x\| = 1$, and any $n \geq 1$, we may write $x = u_n + v_n$ where $u_n \in \langle x_1, \ldots, x_n \rangle$ and $v_n \in \langle x_1, \ldots, x_n \rangle^{\perp}$, and $\|u_n\|^2 + \|v_n\|^2 = 1$. Let T_n denote the finite rank operator

$$T_n = \sum_{i=1}^{n} (x_i, \cdot) \, V x_i$$

Then as $T_n x_j = V x_j$ for $1 \leq j \leq n$

$$Vx - T_n x = V v_n - T_n v_n = V v_n$$

and so

$$\|(V - T_n)x\| = \|V v_n\| \leq a_n \|v_n\| \leq a_n$$

Thus $T_n \to V$, which proves the result. $\square$

Remark 99.1

In a classic paper (P. Enflo, A counterexample to the approximation problem in Banach space, Acta Math. 130 #1 (1.4.54), 309-317) Enflo showed that Theorem 97.1 fails in a general B-space. In fact, Enflo's example shows that the theorem may fail even if the space is separable and reflexive.

<u>Remark 100.1</u> By Thm 73.4, $\sigma(V)$ consists of a countable set, with 0 as the only accumulation point. If $V = V^*$ and $\mathcal{H}$ is not separable, then 0 is necessarily an eigenvalue of (uncountably) infinite multiplicity. This is because, as we will see below, a self-adjoint compact operator has a complete orthonormal basis of eigenvectors.

<u>Lemma 100.2</u> If $V = V^*$ is a self-adjoint, bded operator in a Hilbert space $\mathcal{H}$, then

$$(100.3) \qquad \|V\| = \sup_{\|x\| \le 1} |(x, Vx)| = \sup_{\|x\|=1} |(x, Vx)|$$

<u>Proof</u>: Let $\gamma = \sup_{\|x\|\le 1} |(x, Vx)|$. Clearly $\gamma \le \|V\|$.

Now for any $x, y \in \mathcal{H}$,

$$\operatorname{Re}(x, Vy) = \frac{1}{4}\left[(x+y, V(x+y)) - (x-y, V(x-y))\right]$$

and so

$$|\operatorname{Re}(x, Vy)| \le \frac{\gamma}{4}\left(\|x+y\|^2 + \|x-y\|^2\right) = \frac{\gamma}{2}\left(\|x\|^2 + \|y\|^2\right)$$

Then replacing $x \to e^{i\theta}x$ for appropriate real θ, we conclude that

$$|(x, Vy)| \le \frac{\gamma}{2}\left(\|x\|^2 + \|y\|^2\right)$$

Then replacing x by $\frac{1}{c}x$, and y by cy, any $c>0$, and minimizing over c, we obtain

$$|(x, Vy)| \le \gamma \|x\| \|y\| \qquad \forall x, y \in \mathcal{H}$$

which implies in turn that $\|V\| \le \gamma$. This completes the proof of the lemma. $\square$.

Remark 101.1 Note that (100.3) may fail in the non-self adjoint case, even for 2×2 matrices. For example, for $V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq V^*$

$$\|V\| = \sup_{\|x\|=1} \left\| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \sup_{x_1^2 + x_2^2 = 1} |x_2| = 1.$$

But

$$|(x, Vx)| = \left| \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \right) \right| = |x_1 x_2| \le \frac{1}{2}(x_1^2 + x_2^2) \le \frac{1}{2}$$

Criterion 102.1

Let $k(x,y)$ be a continuous function for

$$-\infty < a \le x, y \le b < \infty .$$

Then the integral operator

$$(Vf)(x) \equiv \int_a^b k(x,y)\, f(y)\, dy$$

is compact in $X = C([a,b])$, the continuous functions on $[a,b]$ with supremum norm.

Proof: Let $M = \max_{a \le x, y \le b} |k(x,y)|$. Then

Then $\|Vf\| \le (b-a)M$ and so $V \in \mathcal{L}(X)$. Suppose $\{f_n\} \subset X$ and $\|f_n\| \le 1$ $\forall n$. Then

$$\|Vf_n\| \le (b-a)M$$

and for $x, y \in [a,b]$

$$|Vf_n(x) - Vf_n(y)| \le \int_a^b |k(x,s) - k(y,s)|\, |f_n(s)|\, ds$$

$$\le \int_a^b |k(x,s) - k(y,s)|\, ds \equiv h(x,y).$$

But as $K(x,y)$ is uniformly cont. on $[a,b]\times[a,b]$

$$\lim_{\delta\downarrow 0} \sup_{|x-y|<\delta} |h(x,y)| = 0.$$

It follows that $\{Vf_n\}$ is an equibounded, equicontinuous family of functions in X, and the result now follows from the Arzela - Ascoli Th^m. $\square$

• Exercise Use Th^m 96.2 to give another proof of Criterion 102.1.

Criterion 103.1 Let $(M, \mathcal{A}, d\mu)$ be a measure space and let $K: M\times M \to \mathbb{C}$ taking $(x,y)\to K(x,y)$ be measurable. Suppose that

(103.2) $$\|K\|_2^2 \equiv \int_{M\times M} |K(x,y)|^2 d\mu(x)\,d\mu(y) < \infty.$$

Then the operator V defined by

(103.3) $$Vf(x) = \int_M K(x,y) f(y)\, d\mu(y), \quad f\in L^2(M,\mathcal{A},d\mu)$$

is compact in $X = L^2(M,\mathcal{A},d\mu)$. Such operators with kernels satisfying (103.2) are called Hilbert-Schmidt operators.

Proof (following [Yosida]) Note first that

$$\|Vf\|^2 = \int \left| \int K(x,y) f(y)\, d\mu(y) \right|^2 d\mu(x)$$

$$\leq \|K\|_2^2 \|f\|^2$$

and so $V \in \mathcal{L}(\mathcal{H})$. By Th^m 96.2, it is enough to show that if $\{f_n\} \subset \mathcal{H} = L^2(\mu)$ is a sequence which converges weakly to $f \in \mathcal{H}$, then Vf_n converges strongly to Vf. By uniform bddedness, $\|f_n\| \leq c$ for some $c < \infty$. Now by the Fubini Theorem, $\int_M |K(x,y)|^2 d\mu(y) < \infty$ for μ-a.e. x. Hence, for such x, $\lim_{n\to\infty} Vf_n(x) = \lim_{n\to\infty} \int K(x,y) f_n(y)\, d\mu(y)$

$$= \lim_{n\to\infty} (K(x,\cdot), f_n)$$

$$= (K(x,\cdot), f)$$

$$= Vf(x)$$

On the other hand, for x as above,

$$|Vf_n(x)|^2 \leq c \int |K(x,y)|^2 d\mu(y) \in L^1(M)$$

The result now follows by the Lebesgue dominated conv. Th^m. $\square$

Criterion 105.1

For functions $f, g : \mathbb{R}^n \to \mathbb{C}$, the operator

$$V = f(x)\, g(p)$$

denotes multiplication by f in x-space and multiplication by g in Fourier space, i.e.,

$$(Vh)(x) = f(x) \int_{\mathbb{R}^n} e^{ix\cdot p} g(p)\, \hat{h}(p) \frac{dp}{(2\pi)^{n/2}},$$

where $\hat{h}(p) = \int_{\mathbb{R}^n} e^{-ip\cdot x} h(x) \frac{dx}{(2\pi)^{n/2}}$ is the Fourier transform of h. If $g \in L^q(\mathbb{R}^n)$, $1 \le q \le 2$, then V has a kernel $K(x,y)$ given by

(105.2) $$K(x,y) = \frac{1}{(2\pi)^{n/2}} f(x)\, \check{g}(x-y)$$

where $\check{g}(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot p} g(p)\, dp$ is the inverse Fourier transform of g. (Recall that the Fourier transform, and similarly the inverse Fourier transform, maps $L^q(\mathbb{R}^n)$, $1 \le q \le 2$, boundedly into $L^p(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$.)

Let $\mathcal{H} = L^2(\mathbb{R}^n)$.

(106.1) If $f, g \in L^\infty(\mathbb{R}^n)$, then $V \in \mathcal{L}(\mathcal{H})$ and
$$\|V\| \le \|f\|_\infty \|g\|_\infty$$

(106.2) If $f, g \in L^2(\mathbb{R}^n)$, then V is Hilbert-Schmidt.

(106.3) If $f, g \in L^\infty(\mathbb{R}^n)$, and $f(x), g(x) \to 0$ as $|x| \to \infty$, then $V \in K(\mathcal{H})$.

Proof: As the Fourier transform is a unitary map in $L^2(\mathbb{R}^n)$, we have

(106.4)
$$\begin{aligned}\|Vh\|_{L^2} &= \|f(g\hat{h})^\vee\|_{L^2} \le \|f\|_\infty \|(g\hat{h})^\vee\|_{L^2} \\ &= \|f\|_\infty \|g\hat{h}\|_{L^2} \le \|f\|_\infty \|g\|_\infty \|\hat{h}\|_{L^2} \\ &= \|f\|_\infty \|g\|_\infty \|h\|_{L^2},\end{aligned}$$

which proves (106.1). The fact that V is Hilbert Schmidt follows directly from (105.2) and the definition (103.2). Finally, suppose that $f(x), g(x) \to 0$ as $|x| \to \infty$.

Then given $\varepsilon > 0$, choose R so that $|f(x)|, |g(x)| < \varepsilon$ if $|x| \leq R$. Let χ_R be the characteristic function of the set $\{|x| < R\}$. Then

$$V = f(x) g(p) = (1 - \chi_R(x)) f(x) g(p) + \chi_R(x) f(x) \chi_R(p) g(p)$$
$$+ \chi_R(x) f(x) (1 - \chi_R(p)) g(p)$$

By (106.1), the first and third terms have norms less than $\varepsilon \|g\|_\infty$ and $\varepsilon \|f\|_\infty$ resp. On the other hand, the 2nd term corresponds to a Hilbert-Schmidt, and hence compact operator. As $\varepsilon > 0$ is arbitrary, it follows by Theorem 67.1 (3), that V is compact. $\square$

<u>Remark 107.1</u> Note that (106.3) implies <u>Rellich's Criterion</u> for compactness in $L^2(\mathbb{R}^n)$: Suppose $F(x)$, $G(x)$ are real valued functions with $F(x), G(x) \geq 1$ and $F(x), G(x) \to \infty$ as $|x| \to \infty$. Then the set

(108.1) $$K_{F,G} = \{ f \in L^2 : \int |f(x)|^2 F^2(x)\, dx \leq 1,\ \int |\hat{f}(p)|^2 G^2(p)\, dp \leq 1 \}$$

is a compact subset of $L^2(\mathbb{R}^n)$.

Indeed, given $\varepsilon > 0$ $\exists$ $R > 0$ st $F(x) > \varepsilon^{-1}$ for $|x| > R$.

Let χ_R be the charac. func. of the set $\{ |x| < R \}$. Then uniformly for $f \in K_{F,G}$, we may write $f = \chi_R f + \hat{\varepsilon}$,

where $\|\hat{\varepsilon}\|_{L^2} \leq \varepsilon$. As $f(x) = \left[\chi_R(x) \frac{1}{G(p)} (G(p) f) \right] + \hat{\varepsilon}$

and $\varepsilon > 0$ is arbitrary, compactness follows from (106.3).

Rellich's criterion implies, in particular, that the unit ball in Sobolev space $H^1 = \{ f : \int |f(x)|^2 dx < \infty, \int |\nabla f|^2 dx < \infty \}$

is locally compact in L^2 i.e. if $\int |f_n|^2 dx \leq 1$ and $\int |\nabla f_n|^2 dx \leq 1$, then $\{f_n\}$ has a subsequence that converges in $L^2(|x| < R)$ for all $R < \infty$. (Why?)

We now come to the key objects of this course, viz.,

Fredholm Operators

Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$.

We say that T is a <u>Fredholm operator</u> if

$$\text{(109.1)} \qquad \dim \ker T < \infty \quad \text{and} \quad \operatorname{codim} T < \infty .$$

By Theorem 73.4 and 84.2, any operator $T \in \mathcal{L}(X)$ of the form $1-K$ for $K \in \mathcal{K}(X)$, is Fredholm. For a general Fredholm operator T we define the <u>index</u> of T by (cf 94.1)

$$\text{(109.2)} \qquad \operatorname{ind}(T) \equiv \dim \ker T - \operatorname{codim} T ,$$

By Theorem 94.2, $\operatorname{ind} T = 0$ if $T = 1 - V$, $V \in \mathcal{K}(X)$.

The following result is basic

<u>Lemma 109.3</u>
Suppose $T \in \mathcal{L}(X, Y)$ and $\operatorname{codim} T < \infty$. Then $\operatorname{ran} T$ is closed.

<u>Proof</u>: Let $\{u_1, \dots, u_n\}$ be a basis for $Y / \operatorname{ran} T$, $n = \operatorname{codim} T < \infty$. For each $i \in \{1, \dots, n\}$, choose $y_i \in u_i$. Then any $y \in Y$ has a unique representation

$$y = \lambda_1 y_1 + \dots + \lambda_n y_n + r$$

where $r \in \operatorname{ran} T$ and $\lambda_i \in \mathbb{C}$. Extend $X \to \tilde{X} = X + \mathbb{C}^n$ and define $\tilde{T} \in \mathcal{L}(\tilde{X}, Y)$ by

$$\tilde{T}(x, \lambda_1, \dots, \lambda_n) = Tx + \sum_{i=1}^{n} \lambda_i y_i$$

Clearly $\tilde{T}$ is surjective and hence by the open mapping Theorem, $\tilde{T}$ must take the open set

$$O = \{(x, \lambda_1, \dots, \lambda_n) : x \in X, \ \sum_{i=1}^{n} |\lambda_i| > 0\}$$

onto an open set $\tilde{T}O$ in Y. But the complement of $\tilde{T}O$ is clearly just $\operatorname{ran} T$. We conclude that $\operatorname{ran} T$ is closed. $\square$

<u>Remark 110.1</u>

If U is a general subspace of Y with $\dim(Y/U) < \infty$, U may not be closed (<u>Exercise</u>). It is a somewhat curious fact that if in addition $U = \operatorname{ran} T$ for some $T \in \mathcal{L}(X, Y)$, then U is closed.