Lecture 9

Two bded operators $T : X \to Y$ and $S : Y \to X$ are said to be <u>pseudoinverses</u> of each other if

(111.1) $$ST = I + K \quad \text{and} \quad TS = I + H$$

where $K \in \mathcal{K}(X)$ and $H \in \mathcal{K}(Y)$

<u>Remark 111.2</u>

To show that S is a pseudoinverse of T it is enough to show $ST = I + K$, $K \in \mathcal{K}(X)$ and $TR = I + L$ for some (other) $R \in \mathcal{L}(Y, X)$ and $L \in \mathcal{K}(Y)$. For then $STR = S + SL$ and so $(I + K)R = S + SL$, or $R = S + M$, where $M = SL - KR \in \mathcal{K}(Y, X)$. Hence

$$TS = TR - TM = I + L - TM$$ and

$H \equiv L - TM \in \mathcal{K}(Y)$. Of course, similar arguments also show that R is a pseudoinverse for T. Clearly pseudoinverses, if they exist, are not unique.

The notion of pseudoinverse provides a very useful characterization of Fredholm operators.

Thm 112.1 An operator $T \in \mathcal{L}(X,Y)$ is Fredholm if and only if it has a pseudoinverse $S \in \mathcal{L}(Y,X)$.

Proof: Suppose S is a pseudoinverse for T. Then

$$ST = 1 + K \quad \text{and} \quad TS = 1 + H$$

for some $K \in \mathcal{K}(X)$ and $H \in \mathcal{K}(Y)$. The first relation shows that $\ker T \subset \ker(1+K)$ and the second relation shows that $\operatorname{codim} T \leq \operatorname{codim} TS = \operatorname{codim}(1+H)$. Thus $\dim \ker T \leq \dim \ker(1+K) < \infty$ and $\operatorname{codim} T \leq \operatorname{codim}(1+H)$ $= \dim \ker(1+H') < \infty$, and so T is Fredholm.

Conversely, suppose that T is Fredholm. Then $\ker T$ is finite dimensional and so by Proposition 60.3 there exists a closed subspace $U \subset X$ which complements $\ker T$, i.e. $X = \ker T \oplus U$.

As T has finite co-dimension it follows by Lemma 109.3 that $\operatorname{ran} T$ is closed, and again by Proposition 60.3, $\operatorname{ran} T$ can be complemented, i.e., $Y = \operatorname{ran} T \oplus V$, where $\dim V = \operatorname{codim} T < \infty$. Now the restriction of T to U is a bijection onto the (closed) subspace $\operatorname{ran} T$, and hence, by the open mapping Theorem, has a bded inverse $L : \operatorname{ran} T \to U$. Set

(113.1) $$\begin{aligned} Su &= Lu \quad , \quad u \in \operatorname{ran} T \\ &= 0 \quad , \quad u \in V \end{aligned}$$

Clearly $S \in \mathcal{L}(Y, X)$. We find

(113.2) $$ST = 1 - P \quad \text{and} \quad TS = 1 - Q$$

where P is the projection in X onto $\ker T$ and along U, and Q is the projection in Y onto V and along $\operatorname{ran} T$. As finite rank operators, P and Q are compact, the above relations prove (111.1). $\square$

Note that Thm 112.1 shows, in particular, that if an operator T is Fredholm, any pseudoinverse of T is also Fredholm.

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The index of Fredholm operators also has an important multiplicative property, first proved by Atkinson and Gohberg. Let L_0, L_1 and L_2 be B-spaces.

<u>Theorem 114.1</u>

Let $A \in \mathcal{L}(L_0, L_2)$ and $B \in \mathcal{L}(L_1, L_0)$ be Fredholm. Then their product $AB \in \mathcal{L}(L_1, L_2)$ is Fredholm and

(114.2) $$\operatorname{ind} AB = \operatorname{ind} A + \operatorname{ind} B.$$

<u>Remark 114.3</u>

We will give the so-called "Russian proof" (see A. A. Kirillov and A. D. Gvishiani, Theorem and Problems in Functional Analysis, Springer-Verlag, New York Heidelberg Berlin, 1982.) This proof of this result is more direct and "elementary" than the proofs commonly given in Western textbooks, see e.g. [Lax].

~~Proof: Let $A = \begin{pmatrix} R & 0 \\ 0 & T \end{pmatrix}$, which is a Fredholm map from $Y \oplus X$ to $Z \oplus Y$. Set $B = \begin{pmatrix} R & 0 \\ 1_Y & T \end{pmatrix}$, where 1_Y is the identity map in Y. We show that B is also Fredholm and~~

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The index of Fredholm operators has important additive properties:

Theorem 114+.1

(Dieudonne) (a) $\operatorname{ind}(T)$ is locally constant in the sense that if $T \in \mathcal{L}(X,Y)$ is Fredholm, then for some $\varepsilon > 0$, $T+S$ is Fredholm for $S \in \mathcal{L}(X,Y)$, $\|S\| \le \varepsilon$, and

(114+.2) $$\operatorname{ind}(T+S) = \operatorname{ind} T$$

(b) $\operatorname{ind}(T)$ is invariant under translation by elements of $K(X,Y)$, i.e. if $T \in \mathcal{L}(X,Y)$ is Fredholm and $S \in K(X,Y)$, then $T+S$ is Fredholm and

(114+.3) $$\operatorname{ind}(T+S) = \operatorname{ind} T$$

(From Riesz-Schauder theory, we already know (114+.3) for $T = 1_X$ and $S \in K(X)$.)

We first prove a useful technical lemma.

Suppose that L_1 and L_2 are Banach spaces with direct sum decompositions into closed subspaces,

(115.1) $$L_i = N_i \oplus M_i, \quad i = 1, 2.$$

Then any operator $T \in \mathcal{L}(L_1, L_2)$ can be written in the form of a 2×2 operator matrix

(115.2) $$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \mathcal{L}(N_1, N_2)$, $B \in \mathcal{L}(M_1, N_2)$, $C \in \mathcal{L}(N_1, M_2)$, $D \in \mathcal{L}(M_1, M_2)$.

<u>Lemma 115.3</u> If T is a Fredholm operator decomposed as in (115.2) above, and D is invertible, then $A - BD^{-1}C$ is Fredholm from $N_1 \to N_2$

(115.4) $$\operatorname{ind} T = \operatorname{ind}(A - BD^{-1}C)$$

<u>Proof</u>: Observe that multiplication of T from the left or the right by an invertible operator S, say, does not change

dimension of the kernel or the codimension i.e.

$$\dim \ker ST = \dim \ker TS = \dim \ker T$$
$$\operatorname{codim} ST = \operatorname{codim} TS = \operatorname{codim} T$$

In particular, if T is Fredholm, then ST and TS are Fredholm and

(116.1) $$\operatorname{ind} ST = \operatorname{ind} TS = \operatorname{ind} T$$

(Check this!) Hence

$$\operatorname{ind}(T) = \operatorname{ind}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{ind}\Bigg[\underbrace{\begin{pmatrix} 1_{N_2} & -BD^{-1} \\ 0 & 1_{M_2} \end{pmatrix}}_{\in \mathcal{L}(L_2, L_2)} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \underbrace{\begin{pmatrix} 1_{N_1} & 0 \\ -D^{-1}C & 1_{M_1} \end{pmatrix}}_{\in \mathcal{L}(L_1, L_1)}\Bigg]$$

invertible!

$$= \operatorname{ind}\left(\begin{pmatrix} A - BD^{-1}C & B - BD^{-1}D \\ C & D \end{pmatrix} \begin{pmatrix} 1_{N_1} & 0 \\ -D^{-1}C & 1_{M_1} \end{pmatrix} \right)$$

$$= \operatorname{ind}\begin{pmatrix} A - BD^{-1}C & 0 \\ C - DD^{-1}C & D \end{pmatrix}$$

$$= \operatorname{ind}\begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}$$

Hence $A - BD^{-1}C$, and D, are Fredholm and

$$\operatorname{ind} T = \operatorname{ind}(A - BD^{-1}C) + \operatorname{ind} D$$

$$= \operatorname{ind}(A - BD^{-1}C) \quad \text{as } D \text{ is invertible} \quad \square$$

We now <u>prove Th^m 114+.1(a)</u>: Suppose $T_0 \in \mathcal{L}(L_1, L_2)$ is Fredholm. Let $N_1 = \ker T_0$ and $M_2 = \operatorname{ran} T_0$. Then as $\dim \ker T_0 < \infty$ and $\operatorname{codim} T_0 < \infty$ it follows $\boxtimes$ (recall $\operatorname{ran} T_0$ is closed by Lemma 109.3) that N_1, M_2 can be complemented in L_1 and L_2 respectively

(117.1) $$L_1 = N_1 \oplus M_1 \quad , \quad L_2 = N_2 \oplus M_2$$

where M_1 and N_2 are closed subspaces. Then T_0 has the form $\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$. Now $D_0 \in \mathcal{L}(M_1, M_2)$ is invertible. Indeed if $m_1 \in M_1$, then $T_0 m_1 \in M_2$.

But $T_0 m_1 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}\begin{pmatrix} 0 \\ m_1 \end{pmatrix} = \begin{pmatrix} B_0 m_1 \\ D_0 m_1 \end{pmatrix}$. Hence as $T_0 m_1$ and $D_0 m_1 \in M_2$, we must have $B_0 m_1 \in M_2$.

But we also have $B_0 m_1 \in N_2$. Hence $B_0 m_1 = 0$ i.e. $T_0 m_1 = D_0 m_1$. But T_0 is 1-1 from M_1 onto M_2.

Hence D_0 is invertible, as claimed. ~~By the above Lemma~~ Any operator

T sufficiently close to T_0 has the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

where $D \in \mathcal{L}(M_1, M_2)$ is invertible (why?). By the above

Lemma, ind T depends only on the dim N_1 and dim N_2

this is because $\dim N_1 < \infty$ and $\dim N_2 = \operatorname{codim} T_0 < \infty$,

and so $A - BD^{-1}C$ is a mapping from a finite dimensional

space to a finite dimensional space i.e. $A - BD^{-1}C$

is isomorphic to a matrix acting from $\mathbb{C}^{\dim N_1}$ to $\mathbb{C}^{\dim N_2}$. For a matrix $J : \mathbb{C}^k \to \mathbb{C}^\ell$.

$\dim \ker J = k - \operatorname{rank} J$ and $\operatorname{codim} J = \dim \ker J' = \ell - \operatorname{rank} J'$

$= \ell - \operatorname{rank} J$. Hence $\operatorname{ind} J = (k - \operatorname{rank} J) - (\ell - \operatorname{rank} J)$

$$= k - \ell.$$

Hence ind T = $\dim N_1 - \dim N_2$, which shows that ind T is locally constant. This proves Th^m 114+.1(a). Note that we have proved that T is Fredholm en route: we could of course prove this directly using the pseudo-inverse for T_0 (Exercise)

We now prove Th^m 114+.1(b): Suppose that $T \in \mathcal{L}(L_1, L_2)$

is Fredholm, $K \in K(L_1, L_2)$. The function $\phi(t) = \operatorname{ind}(T + tK)$

is defined on the whole line. Indeed if S is a pseudoinverse of T, $TS = 1 + P$, $ST = 1 + Q$, P, Q compact, then

$$(T+tK)S = TS + tKS = 1 + \underbrace{(P + tKS)}_{\text{compact}}$$

$$S(T+tK) = ST + tSK = 1 + \underbrace{(Q + tSK)}_{\text{compact}}$$

so S is also a pseudoinverse for $T+tK$: thus $T+tK$ is Fredholm $\forall t \in \mathbb{R}$. However by Th^m 114.1(a), $\operatorname{ind}(T+tK)$ is locally constant and so $\operatorname{ind} T = \phi(0) = \phi(1) = \operatorname{ind}(T+K)$ as $\mathbb{R}$ is connected. This proves Th^m 114.1(b)

Finally we <u>prove Th^m 114.1</u>. Consider the auxiliary operator $A \oplus B$ acting from $L_0 \oplus L_1$ to $L_2 \oplus L_0$. Clearly $\operatorname{ind}(A \oplus B) = \operatorname{ind} A + \operatorname{ind} B$. Further, for sufficiently small ε,

$$\operatorname{ind} A + \operatorname{ind} B$$

$$= \operatorname{ind}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \operatorname{ind}\begin{pmatrix} A & 0 \\ \varepsilon 1_{L_0} & B \end{pmatrix}, \quad \text{by Th}^m \text{ 114.1(a)}$$

$$= \operatorname{ind}\left[\begin{pmatrix} 1_{L_2} & -\varepsilon^{-1}A \\ 0 & 1_{L_0} \end{pmatrix}\begin{pmatrix} A & 0 \\ \varepsilon 1_{L_0} & B \end{pmatrix}\begin{pmatrix} \varepsilon^{-1} 1_{L_0} & B \\ 0 & -\varepsilon 1_{L_1} \end{pmatrix}\right]$$

$$= \operatorname{ind}\left(\begin{pmatrix} A-A & -\varepsilon^{-1}AB \\ \varepsilon 1_{L_0} & B \end{pmatrix}\begin{pmatrix} \varepsilon^{-1}1_{L_0} & B \\ 0 & -\varepsilon 1_{L_1} \end{pmatrix}\right)$$

$$= \operatorname{ind}\begin{pmatrix} 0 & AB \\ 1_{L_0} & \varepsilon B - \varepsilon B \end{pmatrix}$$

$$= \operatorname{ind}\begin{pmatrix} 0 & AB \\ 1_{L_0} & 0 \end{pmatrix} = \operatorname{ind} AB .$$ This proves Th^m 114.1.

Note that we have proved that AB is Fredholm en route to proving (114.2): we could also have proved this directly (as in Th^m 114.1 (a)) using the pseudoinverses of A and B (exercise). $\square$.

Corollary 120.1

Suppose $T \in \mathcal{L}(X,Y)$ is Fredholm with pseudo-inverse S. Then

(120.2) $$\operatorname{ind} T = -\operatorname{ind} S$$

Proof: Apply Theorems 114.1 and Th^m 94.2 to $TS = 1 + K$, $K \in \mathcal{K}(Y)$. $\square$

Theorem 120.3

An operator $T \in \mathcal{L}(X,Y)$ is Fredholm if and only if its dual $T' \in \mathcal{L}(Y',X')$ is Fredholm and

(120.4) $$\operatorname{ind} T' = -\operatorname{ind} T$$

<u>Remark</u> Relation (120.4) is well-known for matrices!

<u>Proof of Th^m 120.3</u> If T is Fredholm, ran T is closed and (84.2) (84.3) hold in the sense of Remark 84.4

i.e. $\operatorname{codim} T = \dim \ker T'$ and $\operatorname{codim} T' = \dim \ker T$

Thus $\dim \ker T' = \operatorname{codim} T < \infty$ and $\operatorname{codim} T' = \dim \ker T < \infty$

so that T is Fredholm, and (120.4) follows. Conversely, if T' is Fredholm, then ran T' is closed, and hence ran T is closed, and so (84.2) (84.3) again hold. Thus

$$\dim \ker T = \operatorname{codim} T' < \infty \quad \text{and} \quad \operatorname{codim} T = \dim \ker T' < \infty$$

which $\Rightarrow$ T is Fredholm and (120.4) follows as before. $\square$

Although the index is stable under small perturbations, $\dim \ker T$ and $\operatorname{codim} T$ generally decrease.

<u>Theorem 121.1</u>
Suppose $T \in \mathcal{L}(X, Y)$ is Fredholm. Then for $M \in \mathcal{L}(X, Y)$ with small norm,

(122.1) $\dim \ker (T+M) \le \dim \ker T$ and $\operatorname{codim}(T+M) \le \operatorname{codim} T$.

Proof: Choose S as in (113.2) so that $ST = 1 - P$, $\dim \operatorname{ran} P = n = \dim \ker T < \infty$. Then for $\|M\| < \|S\|^{-1}$,

$$S(T+M) = (1+SM)(1-(1+SM)^{-1}P)$$

and hence $\dim \ker (T+M) \le \dim \ker (1-(1+SM)^{-1}P)$.

Now $x \in \ker(1-(1+SM)^{-1}P) \Rightarrow x = (1+SM)^{-1}Px \in \operatorname{ran}(1+SM)^{-1}P$. But as $\dim \operatorname{ran} P = n$, $\dim \operatorname{ran}(1+SM)^{-1}P = n$.

Thus $\dim \ker (T+M) \le n = \dim \ker T$. As $\operatorname{ind}(T+M) = \operatorname{ind} T$, it follows that $\operatorname{codim}(T+M) \le \operatorname{codim} T$. $\square$

The next result extends the relationship between AB and BA to Fredholm operators. The proof is immediate from Th^m 90.1.

Theorem 122.2

Let X and Y be B-spaces and let $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$. Suppose $\lambda \ne 0$. Then

(122.3) $\lambda + AB$ is Fredholm $\Leftrightarrow$ $\lambda + BA$ is Fredholm

(122.4) $\operatorname{ind}(\lambda + AB) = \operatorname{ind}(\lambda + BA)$. $\square$