Problem Set #10
Fall 2015

Functional Analysis
due: Nov 19, 2015

1. Let $h(\theta) = 1 - \cos\theta$ on $S^1 = \{e^{i\theta} : 0 \le \theta \le 2\pi\}$ and let $T_h f = P_+ h f$ be the associated Toeplitz operator on H_+. Show that T_h is not Fredholm (Hint: Show that the functions $g(z) = \frac{1}{z-a}$, $|a|>1$, are not in ran T_h)

Remark: This problem illustrates the general fact that if $T_h = P_+ h$ on H_+ is Fredholm, then $h^{-1} \in L^\infty(S^1)$ (See Litvinchuk - Spitkovskii, Factorization of measurable matrix functions, Oper. Th^y Vol. 25, Birkhauser : Th^m 3.13, p111.)

2. Let $a, b, c, d \in \mathbb{C} \setminus S^1$ where $S^1 = \{|z| = 1\}$. For $z \in S^1$, let

$$h_1(z) = \frac{z-a}{z-b} \qquad h_2(z) = \frac{z-a}{(z-b)(z-c)}$$

$$h_3(z) = \frac{(z-a)(z-d)}{(z-b)} \qquad h_4(z) = \begin{pmatrix} (z-a)^{-1} & (z-b)^{-1} \\ (z-c)^{-1} & (z-d)^{-1} \end{pmatrix}$$

and let $T_{h_j} f = P_+ h_j f$, $f \in H_+$, $j = 1, \dots, 4$ be the associated Toeplitz operators.

(i) Show that for $j = 1, 2, 3$, T_{h_j} is Fredholm and compute $\operatorname{ind} T_{h_j}$, $\dim \ker T_{h_j}$ and $\operatorname{codim} T_{h_j}$ in each case

(ii) Show that if $|a+d-b-c| \neq |ad-bc|$, T_{h_4} is Fredholm and compute $\operatorname{ind} T_{h_4}$, $\dim \ker T_{h_4}$ and $\operatorname{codim} T_{h_4}$. If $|a+d-b-c| = |ad-bc|$, show that T_{h_4} is not Fredholm.

3. (challenging problem!) Let

$$h(z) = e^{i\pi/4}, \quad z = e^{i\theta}, \quad 0 < \theta < \pi$$

$$= e^{-i\pi/4}, \quad z = e^{i\theta}, \quad \pi < \theta < 2\pi$$

Show that the associated Toeplitz operator T_h is invertible in H_+. Thus, although $h(z)$ is not continuous on $S^1 = \{|z| = 1\}$, T_h is Fredholm of index $= 0$ and $\dim \ker T_h = 0$ and $\operatorname{codim} T_h = 0$. (Hint: factorize $h(z)$ in the following way: $\exists\ \delta(z)$ analytic in $\mathbb{C}\setminus S^1$ st.

- $\delta_+(z) = \delta_-(z)\, h(z)$, $z \in S^1$

  where $\delta_+(z) = \lim\limits_{\substack{z' \to z \\ |z'|<1}} \delta(z')$

  and $\delta_-(z) = \lim\limits_{\substack{z' \to z \\ |z'|>1}} \delta(z')$

- $\delta(z) \to 1$ as $z \to \infty$.

Then use δ to solve $T_h f_+ = g_+$ for $f_+ \in H_+$, given $g_+ \in H_+$.)

4. Let
$$h(\theta) = \begin{cases} e^{i\pi/2}, & 0<\theta<\pi \\ e^{-i\pi/2}, & \pi<\theta<2\pi \end{cases}$$

Show that the associated Toeplitz operator T_h is not invertible. Is it still Fredholm?