Problem Set #12
Fall 2015

Functional Analysis
due : Dec 10, 2015

1. Let $\mathcal{H} = \ell^2(-\infty, \infty) = \{a = \{a_n\}_{n=-\infty}^{\infty} : \sum_{-\infty}^{\infty} |a_n|^2 < \infty\}$,

Let $(Aa)_n = a_{n+1} + a_{n-1}$, $a \in \mathcal{H}$. Show that a is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\mu_1) \oplus L^2(\mathbb{R}, d\mu_2)$ where μ_1 and μ_2 have support on $[-2, 2]$. (Hint : use Fourier Theory)

2. Show that if $A \in \mathcal{B}_1(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$ then

$$\|BA\|_1 \le \|B\| \, \|A\|_1$$

and

$$\|AB\|_1 \le \|B\| \, \|A\|_1$$

(Hint : use min-max)

3. Show by example that if $\sum_{n=1}^{\infty} |(\varphi_n, A\varphi_n)| < \infty$ for some $A \in \mathcal{L}(\mathcal{H})$ and $\{\varphi_n\}$ is some orthonormal basis, that A may not be trace class

4. Show that for a separable Hilbert space $\mathcal{H}$

$$(a) \quad \mathcal{B}_1(\mathcal{H}) = (\mathcal{K}(\mathcal{H}))^*$$

and $$(b) \quad \mathcal{L}(\mathcal{H}) = (\mathcal{B}_1(\mathcal{H}))^*$$

5. For an operator $A \in \mathcal{L}(\mathcal{H})$, let $\Gamma_n(A)$ be the operator on $\otimes^n \mathcal{H}$ with

$$\Gamma_n(A)\, \varphi_1 \otimes \cdots \otimes \varphi_n = A\varphi_1 \otimes \cdots \otimes A\varphi_n$$

Show that $\Gamma_n(A)$ is bded on $\otimes^n \mathcal{H}$.

(***) 6. Suppose P and Q are bounded projections in a Hilbert
(challenging)
space $\mathcal{H}$. Suppose $P-Q \in \mathcal{B}_1(\mathcal{H})$. Then $tr(P-Q)$ is an integer. (Hint: Show that if $\lambda_0 \neq \pm 1$ or 0 is an eigenvalue of $P-Q$ then

algebraic multiplicity of λ_0 = algebraic multiplicity of $-\lambda_0$.

Then use $tr\, A = \sum \lambda_i$ where λ_i are the eig's of A, counting multiplicity. Note that for $A = P-Q$ and $B = P+Q-1$, we have $AB+BA = 0$ and $A^2+B^2 = 1$.)

7. Let A be an $n \times n$ matrix. Show that

$tr\, A = 0$ iff $A = [B,C] = BC - CB$ for some matrices B and C.