Functional Analysis Fall 2015

Problem Set # 3 Due: Thursday, Sep 30

1. Let X be a Banach space.

(a) Show that if $x_1, \dots, x_n$ are independent in X, then there exist $x_1', \dots, x_n' \in X'$ such that

$$\langle x_i', x_j \rangle = \delta_{ij} \quad , \quad 1 \le i, j \le n$$

(b) Show that if $x_1', \dots, x_n'$ are independent in X', then there exist $x_1, \dots, x_n \in X$ such that

$$\langle x_i', x_j \rangle = \delta_{ij} \quad , \quad 1 \le i, j \le n$$

(Hint: Show that

$$N_1 \cap N_2 \cap \dots \cap N_n \subsetneq N_1 \cap N_2 \cap \dots \cap N_{n-1} \subsetneq \dots \subsetneq N_1 \cap N_2 \subsetneq N_1)$$

2. Let T be the operator

$$Tf = \frac{df}{dx}$$

with domain $\tilde{D}(T) = \{ f \in L^2(0,1) : f \text{ is absolutely continuous}, f' = \frac{df}{dx} \in L^2(0, 2\pi) \}$

in $\mathfrak{H} = L^2(0,1)$. Show that T is closed on $\tilde{D}(T)$.

3. Show that the unit ball in an infinite dimensional Hilbert space contains infinitely many <u>disjoint</u> translates of a ball of radius $\sqrt{2}/4$. Conclude that one cannot have a non trivial translation invariant measure on an infinite dimensional Hilbert space.

4. Show that if $T_n \in \mathcal{L}(\mathcal{H})$, $n \geq 0$, for some Hilbert space $\mathcal{H}$, and $\lim_{n\to\infty} (T_n x, y)$ $\exists$ $\forall x, y \in \mathcal{H}$, then $\|T_n\| \leq c$, $\forall n \geq 0$.

5. Let $\{x_n\}_{n=-\infty}^{\infty}$ be a set of vectors in a Hilbert space $\mathcal{H}$, so that $a_{nm} = |(x_n, x_m)|$ is the matrix (in the natural basis) of an operator A on $\ell_2(-\infty, \infty)$. Prove that

$$\sum_{n=-\infty}^{\infty} |(f, x_n)|^2 \leq \|A\| \, \|f\|^2$$

6. Prove that a Banach space X is reflexive if and only if X^* is reflexive. (Hint: If $X \neq X^{**}$ find a bded linear functional on X^{**} which vanishes on X.)

7. Consider the space Y of polynomials as a linear subspace of $C[a,b]$. Define a linear functional f on Y as follows: $f(a_n x^n + \cdots + a_0) = a_0$. Show that f can be extended to a bounded linear functional on $C[a,b]$ if and only if $0 \in [a,b]$.