Functional Analysis Fall 2015

Problem Set #4 Due Thurs Oct 8

1. Show that every subspace Y of a linear space X can be complemented, in the algebraic sense

$$X = Y \oplus \tilde{Y}$$

for some subspace $\tilde{Y} \subset X$

2. Show that complement $\tilde{Y}$ of a closed subspace Y of B-space is not unique, by showing how to modify $\tilde{Y}$ to obtain an infinite number of new complements $\tilde{\tilde{Y}}$, $X = Y \oplus \tilde{\tilde{Y}}$. (Here we mean closed complements $\tilde{Y}$ and $\tilde{\tilde{Y}}$).

3. Let X be a normed linear space. Then

X is complete $\Longleftrightarrow$ X has the following property: if $\sum_{n=1}^{\infty} \|x_n\| < \infty$, $x_n \in X$, then

$\exists x \in X$ st $\sum_{1}^{n} x_j \to x$ in X

In words, the latter property says that "every absolutely summable series, is summable in X^{4}.

4. Not every closed subspace of a Banach space can be complemented. The following example due to R. Phillips shows that $c_0 = \{x = \{x_1, x_2, \ldots\} : \lim_i x_i = 0\}$ cannot be complemented in ℓ^∞. The following argument is taken from R.E. Megginson, An introduction to Banach space Theory, Grad. Texts Math, Springer-Verlag, New York, Berlin, Heidelberg, 1998)

Let A be the disk algebra i.e.

$$A = \{ f(z) : f(z) \text{ analytic in } |z|<1 \text{ and continuous in } |z| \le 1 \}$$

with norm $\|A\| = \sup_{|z|\le 1} |f(z)|$

(a) Show that A may be identified as a B-space with a closed subspace A_Σ of C_Σ, the continuous functions on $\Sigma = \{|z|=1\}$.

(b) Show that $g(\theta) = \sum_{n=2}^{\infty} \frac{\sin n\theta}{n \log n} = \lim_{N\to\infty} \sum_{n=2}^{N} \frac{\sin n\theta}{n \log n}$ lies in C_Σ. Conclude that the stand projection

$$P^s : \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \to \sum_{n=0}^{\infty} a_n e^{in\theta}$$

is not bounded from C_Σ to A_Σ. (Hint: sum

$$\sum_{n=2}^{N} \frac{\sin n\theta}{n \log n}$$ by parts.)

(c) A_Σ cannot be complemented in C_Σ: (Proof following D. J. Newman)

Let P be a bounded projection from C_Σ to A_Σ

i.e.,
- $\|Pf\|_\infty \le K \|f\|_\infty$, $f \in C_\Sigma$, for some $K < \infty$.
- $Pf = f$, $f \in A_\Sigma$

For $f \in C_\Sigma$ define

(3.1) $$\tilde{P}f = \frac{1}{2\pi} \int_{-\pi}^{\pi} (Pf_\theta)_{-\theta} \, d\theta$$

where $f_\theta(\psi) = f(\theta + \psi)$. Note that $\theta \mapsto (Pf_\theta)_{-\theta}$ is continuous, so that the RHS in (3.1) is well-defined as a Riemann integral.

(i) Show that $\tilde{P}$ is a bded map from C_Σ to A_Σ

(ii) Show that

$$\tilde{P}(e^{in\theta}) = e^{in\theta} \quad , n \geq 0$$
$$= 0 \quad , n < 0$$

Thus $\tilde{P}$ is the standard projection P^s, which yields a contradiction.

5. Let X & Y be B-spaces. Suppose that $T_n \in K(X,Y)$, $n \geq 1$, and $T \in \mathcal{L}(X,Y)$ and that $T_n \to T$ in operator norm i.e. $\|T_n - T\| \to 0$ as $n \to \infty$. Show that $T \in K(X,Y)$. Thus the compact operators form a closed set in $\mathcal{L}(X,Y)$.