Problem set #5 — Functional Analysis

Fall 2015

Due: Thursday October 15

1. Show that a linear functional $T$ on a B-space $X$ is bounded if and only if $\ker T$ is closed.

2. Let $T \in \mathcal{L}(X, Y)$, $X$ & $Y$ B-spaces. Then $T$ is compact if and only if $[T] \in \mathcal{L}(X/\ker T, Y)$ is compact.

3. Let $K(x,y)$ be a continuous function on $[0,1]\times[0,1]$. Define

(3.1) $$Tf(x) = \int_0^1 K(x,y)\, f(y)\, dy$$

for any integrable function $f$ on $[0,1]$. Show that for any $1 \le p \le \infty$, $T$ is a compact map from $L^p([0,1], dx)$ to itself.

4. Let $K(x,y) = \max(x,y)$, $0 \le x, y \le 1$, and let $T$ be defined as in (3.1) above. Compute $\sigma(T)$ in

$L^2([0,1], dx)$. What about in $L^p([0,1], dx)$ for any $1 \le p \le \infty$?

5. Let $K(x,y) = 1$ if $0 \le y \le x \le 1$
$= 0$ if $0 \le x < y \le 1$

and define $T$ as in (3.1) above. Show that $T$ is compact in $L^2([0,1], dx)$ and compute $\sigma(T)$.

6. Let $K(x,y) = \dfrac{(x/y)^{\frac{1}{2}} - 1}{x - y}$ on $(0,\infty) \times (0,\infty)$

and define $T$ as in (3.1) above with $(0,1)$ replaced by $(0,\infty)$.

(a) Show that $T \in \mathcal{L}(L^4(0,\infty), dx)$

(Hint: consider $\displaystyle\int_0^\infty \int_0^\infty dx\,dy \frac{g(x)\,f(y)}{\sqrt{y}\,(\sqrt{x}+\sqrt{y})}$
where $f \in L^4(0,\infty)$, $g \in L^{4/3}(0,\infty)$ and change variables $x = r\cos\theta$, $y = r\sin\theta$.)

(b) Show that $T$ is not compact on $L^4(0,\infty)$

(Hint: consider the functions
$f_n(x) = n^{-1/4}$, $n < x < 2n$
$= 0$ otherwise)

7. Let $X = \ell_2 = \{x = (x_1, x_2, \ldots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$

Define $T : X \to X$ by

$$T(x_1, x_2, \ldots) = \left(0, \frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots\right).$$

Show that $T$ is compact but $T$ has no eigenvalues : hence $\sigma(T) = \{0\}$. On the other hand, show that $\lambda = 0$ is an eigenvalue of $T'$. Thus although $\sigma(T) = \sigma(T')$, $\lambda = 0$ is an eigenvalue of $T'$ but not of $T$.

8. Let $K(x, y)$ be measureable on $(0, \infty) \times (0, \infty)$ and set

$$c = \max\left(\sup_{x \geq 0} \int_0^\infty |K(x, y)| dy, \; \sup_{y \geq 0} \int_0^\infty |K(x, y)| dx\right)$$

Let $T$ be defined as in (3.1) above with $(0,1)$ replaced by $(0, \infty)$. Show that if $c < \infty$, $T \in \mathcal{L}(L^q(0, \infty) \to \mathcal{L}(L^q(0, \infty))$, $1 \leq q \leq \infty$, and $\|T\|_{q \to q} \leq c$.

9. Show that eigenvectors corresponding to distinct eigenvalues of an operator $V$ are independent.

10. Show that every operator of form (3.1) with $k(x,y) \in C([0,1] \times [0,1])$, is a norm limit of operators of finite rank.