Problem Set #8 — Functional Analysis

Fall 2015 — due: Nov 5, 2015

1. Suppose X is a B-space, $\dim X = \infty$, and let $A \in K(X)$.

Show that $0 \in \sigma(A)$ and, moreover, if ran A is ∞-dimensional, it is never closed.

2. Give an example of a pair of operators $A, B \in \mathcal{L}(X)$ such that

$$0 \in \sigma(AB) \quad \text{and} \quad 0 \notin \sigma(BA).$$

3. Suppose $T \in \mathcal{L}(X)$ is idempotent i.e. $T^2 = 1_X$.

Show that

(i) $\sigma(T) \subset \{-1, 1\}$

(ii) if $T \neq \pm 1_X$ then 1 and -1 are in $\sigma(T)$, and, moreover, we have the direct sum decomposition

$$X = M_1 \oplus M_{-1}$$

where $M_1 = \ker(T - 1_X)$ & $M_{-1} = \ker(T + 1_X)$

4. Another proof of $\operatorname{ind} RT = \operatorname{ind} R + \operatorname{ind} T$ — (1.1)
when T is Fredholm from $X \to Y$ and R Fredholm from $Y \to Z$ (Th^m 114.1)

Let $T \in \mathcal{L}(X,Y)$ & $R \in \mathcal{L}(Y,Z)$ be Fredholm.

(i) Show that Y has a direct sum decomposition

(2.1) $$Y = Y_0 \oplus Y_1 \oplus Y_2 \oplus Y_3$$

where each T_i is closed, $0 \le i \le 3$, $k_i = \dim Y_i < \infty$, $i = 1,2,3$, and

(2.2)

(a) $Y_0 \oplus Y_1 = \operatorname{ran} T$

(b) $Y_1 \oplus Y_2 = \ker T$

(c) $\operatorname{ran} R = \operatorname{ran} RT \oplus RY_3$

(ii) Use (2.1) to prove (1.1).

5. Show that if A is bdd and self-adjoint in a Hilbert space $\mathcal{H}$, then

$$\sigma(A) = \{0\} \iff A = 0$$

6. Show that if $\mathcal{H}$ is a complex Hilbert space, then for $A \in \mathcal{L}(\mathcal{H})$,

$$(u, Au) \in \mathbb{R} \quad \forall u \in \mathcal{H}$$

$\Rightarrow$ $A = A^*$.