Problem Set #9
Fall 2015

Functional Analysis
due: Nov 12, 2015

1. Show that the Analytic Fredholm Theorem I is equivalent to Riesz-Schauder theory i.e. in the proof of Anal. Fred. Th^m I we used the Riesz-Schauder th^m: conversely, apply Anal. Fred. Th^m I to $f(z) = zK$, K compact, to deduce the facts of Riesz-Schauder Theory.

2. Let $H^+ = \{ \sum_{n=0}^{\infty} a_n e^{in\theta} : \sum_{n=0}^{\infty} |a_n|^2 < \infty \} \subset L^2(\mathbb{T}, \frac{d\theta}{2\pi})$ where $\mathbb{T}$ is the unit circle. Let P^+ denote the orthog. projection of $L^2 \to H^+$ given by

$$P^+\left(\sum_{-\infty}^{\infty} a_n e^{in\theta} \right) = \sum_{n=0}^{\infty} a_n e^{in\theta}$$

Consider the Toeplitz operator T_h from $H^+ \to H^+$

$$T_h f = P^+ h f, \qquad f \in H^+$$

where $h = h(\theta) \in L^\infty(\mathbb{T})$.

Show that

that

(i) $T_h \in \mathcal{L}(H^+)$

(ii) If $h \in C(\mathbb{T})$, $h(\theta) \geq c > 0$, $0 \leq \theta \leq 2\pi$, Then

T_h is Fredholm in H^+

(Hint: Show that $P_+ h P_-$ is compact in $L^2(\mathbb{T}, d\theta/2\pi)$ if $h(\theta)$ is continuous. Here $P_-\left(\sum_{-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{-\infty}^{-1} a_n e^{in\theta}$, i.e.

$P_- = 1 - P_+$)

(iii) Let $h(\theta) = 1 - \cos\theta$ on $\mathbb{T}$. Show that T_h is not

Fredholm (Hint: Show that the functions

$g(z) = (z-a)^{-1}$, $|a| > 1$ are not in ran T_h)

3. Let $A \in \mathcal{L}(X)$, X a B-space. Prove that the set of λ such that λ is in $\sigma(A)$ but not an eigenvalue and ran $(\lambda - A)$ is closed but not all of X, is an open subset of $\mathbb{C}$.

4. If an operator A maps an n-dimensional space X into an m dimensional space Y, then ind $A = n - m$.

5. Let X be a separable Banach space. By a Schauder basis in X, we mean a set $x_n \in X$, $\|x_n\| = 1$, $n = 1, 2, \ldots$, such that for every $x \in X$, there is a unique series

$$\sum_{n=1}^{\infty} a_n x_n$$

such that

$$\lim_{n \to \infty} \left\| x - \sum_{i=1}^{n} a_i x_i \right\| = 0.$$

(1) Show that $a_n = a_n(x)$ is a bounded linear functional of x

(2) Exhibit a Schauder basis for $C[0,1]$, the continuous functions on $[0,1]$ with sup norm.

Remarks (1) Not every separable Banach space has a Schauder basis. This is a result of P. Enflo.

(2) Note also that a Schauder basis very different from a Hamel basis.