1. Use Jordan forms to show that matrices with distinct eigenvalues are dense in the space of all matrices.

   Answer: This result is equivalent to show that any matrix is the limit of a sequence of matrices with distinct eigenvalues.

   Let A be any matrix, J be it's Jordan form, and S be such that $A = SJS^{-1}$. We shall construct a $\tilde{J}_m$ such that $\tilde{J}$ is close to J, but has distinct eigenvalues. To do this, let $\varepsilon = \min\{\min_{k\neq j} |J_{jj} - J_{kk}|, 1\}$, then if we define

$$\tilde{J}_m = \begin{pmatrix} J_{11} + \frac{\varepsilon}{m} & * & & 0 \\ & J_{22} + \frac{2\varepsilon}{m} & * & \\ & & \ddots & \\ 0 & & & J_{nn} + \frac{n\varepsilon}{m} \end{pmatrix} = J + \frac{\varepsilon}{m} \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & \ddots & \\ 0 & & & n \end{pmatrix}.$$

   By the definition of ε, $\tilde{J}_m$ has distinct diagonal entries. Since $\tilde{J}_m$ is triangular, this then gives that it has distinct eigenvalues.

   Let $\tilde{A}_m = S\tilde{J}_m S^{-1} \to A$. Since $\tilde{A}_m$ has the same eigenvalues of $\tilde{J}_m$, which are distinct, $\tilde{A}_m$ has distinct eigenvalues. We also see that

$$\tilde{A}_m - A = S(\tilde{J}_m - J)S^{-1} \to 0 \text{ as } m \to \infty, \text{ since } \tilde{J}_m - J \to 0.$$

   Thus A is the limit of a sequence matrices with distinct eigenvalues. As A is general, we have proved the desired result.

2. Let A be the matrix the complex 3×3 matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{pmatrix}.$$

   Compute the Jordan form for A and show that A is similar to a diagonal matrix if and only if $a = 0$.

   Answer: First we see that

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 & 0 \\ a & 2-\lambda & 0 \\ b & c & -1-\lambda \end{pmatrix} = -(2-\lambda)^2(1+\lambda),$$

   so the eigenvalues of A are -1 and 2. We see that -1 has multiplicity 1, so the Jordan block of that eigenvalue is $B_2 = \begin{pmatrix} 1 \end{pmatrix}$. The eigenvalue 2 has multiplicity 2, so we must check $\text{nullity}(A - 2I)$. We see that

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & -3 \end{pmatrix},$$

   so $\text{nullity}(A - 2I) = 2$ if and only if $a = 0$. Since $\text{nullity}(A - 2I)$ is the number of linearly independent genuine eigenvectors which is equal to the number of Jordan blocks, we have that the Jordan form of A is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \text{ if } a = 0 \text{ and } \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ otherwise.}$$

If $a \neq 0$, we know that A is similar to $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which is not similar to a diagonal matrix. If $a = 0$, the Jordan form is diagonal. Thus A is similar to a diagonal matrix if and only if $a = 0$.

3. Let $a_0, a_1, \ldots, a_n - 1$ be complex numbers and let V be the space of all n times differentiable functions f on an interval of the real line, which satisfy the differential equation

$$D^n f + a_{n-1} D^{n-1} f + \cdots + a_1 Df + a_0 f = 0,$$

where D is the differential operator d/dx. Clearly V is invariant under D. What is the Jordan form for D on V?

Answer:

(a) Let J be the Jordan form of D. Let $q_j =$ the number of Jordan blocks. We know that q_j is equal to the number of genuine eigenvectors of eigenvalue c_j, which is in turn equal to the nullity of $D - c_j$.
To find the nullity of $D - c_j$, suppose f is such that $Df - c_j f = 0$, then f must be of the form $f(x) = Ae^{c_j x}$. Thus the null space of $(D - c_j)$ is one dimensional, and so $d_j = 1$. Thus the J is composed of blocks of the form

$$B_j = \begin{pmatrix} c_j & 1 & \\ & \ddots & 1 \\ & & c_j \end{pmatrix},$$

with distinct c_j's.

(b) To determine the size of these blocks, we define $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ and note that for all $f \in V$,

$$p(D)f = D^n f + a_{n-1} D^{n-1} f + \cdots + a_1 Df + a_0 f = 0.$$

Thus $p(D) = 0$ and so by the definition of the minimal polynomial $m_D(s)$, $m_D(s)$ must divide $p(s)$.

(c) If we factor p as $p(s) = \prod_{j=1}^{k} (s - c_j)^{r_j}$. Then, since roots of m_D are also roots of p, we see that we may write $m_D(s) = \prod_{j=1}^{k} (s - c_j)^{d_j}$, where $0 \leq d_j \leq r_j$.
From the theory of ODEs, we know that the solution space of $p(D)f = 0$ has dimension equal to the degree of p, which is n. We also know that the solution space of $m_D(D)f = 0$ has dimension equal to the degree of m_D. If the degree of m_D was less than n, there would exist an f such that $p(D)f = 0$ but $m_D(D)f \neq 0$, which would contradict the fact that $m_D(D) = 0$.
Thus $n = \deg(m_D) = \sum_{j=1}^{k} d_j$. The upper bound $d_j \leq r_j$ then gives that $d_j = r_j$ for all j and so $m_D = p$. Since we also know that the d_j's, the multiplicity of the roots of the minimal polynomials, are the size of the largest Jordan blocks of each eigenvalue. Since we proved in part a that there is only 1 Jordan block of each eigenvalue, we thus have that

$$J = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}, \text{ where } B_j = \begin{pmatrix} c_j & 1 & \\ & \ddots & 1 \\ & & c_j \end{pmatrix} \text{ is an } r_j \times r_j \text{ matrix.}$$

4. Let $X_1 \subset X_2 \subset \ldots \subset X_k$ be a tower of linear spaces. Show that

$$\dim(X_1) + \dim(X_2/X_1) + \cdots + \dim(X_k/K_{k-1}) = \dim X_k.$$

Answer: We recall that for any two linear spaces X, Y such that $Y \subset X$, we have that

$$\dim(Y) + \dim(X/Y) = \dim(X) \Rightarrow \dim(X/Y) = \dim(X) - \dim(Y).$$

Applying this $k-1$ times, we have that

$$\dim(X_1) + \sum_{j=2}^{k} \dim(X_j/X_{j-1}) = \dim(X_1) + \sum_{j=2}^{k} [\dim(X_j) - \dim(X_{j-1})] = \dim(X_k).$$