1. Let X be the space of polynomials of degree $< n$, and let Y be the set of polynomials that are zero at $t_1, \dots, t_j$, $j < n$, $t_i \in \mathbb{R}$. Determine $\dim(Y)$ and $\dim(X/Y)$.

   Answer: Let $t_1, \dots, t_k$ be the k distinct points out of $t_1, \dots, t_j$. For any $y \in Y$, the fundamental theorem of algebra tells us that we may write $y(t) = g(t) \prod_{i=1}^{k}(t - t_i)$, where g is a polynomial of degree $< n - k$. Since g is a polynomial, it can be written as

   $$g(t) = a_0 + a_1 t + \cdots + a_{n-k-1} t^{n-k-1}$$

   This tells us that if we define $y_l = t^l \prod_{i=1}^{k}(t - t_i)$, then any $y \in Y$ may be written as a linear combination of the y_l's.

   $$y(t) = \sum_{l=1}^{n-k} a_l y_l(t),$$

   so the y_l's span Y. They may also be seen to be linearly independent by having different growth rates as $t \to \infty$, so $\{y_l\}_{l=1}^{n-k}$ is a basis for y. Since this basis has $n - k$ elements, we know that $\dim(Y) = n - k$.

   Since we know that $\dim(X) = n$, Theorem 6 of Chapter 1 of Lax's book tells us that

   $$\dim(X/Y) = \dim(X) - \dim(Y) = k.$$

2. In Theorem 7 of chapter 2 of Lax's book, take the interval I to be $[-1, 1]$, and n to be 3. Choose the 3 points to be $t_1 = -a$, $t_2 = 0$, and $t_3 = a$, $a \in (0, 1]$.

   (a) Determine the weights m_1, m_2, and m_3 such that

   $$\int_I p(t)dt = m_1 p(t_1) + m_2 p(t_2) + m_3 p(t_3), \tag{1}$$

   holds for all polynomials of degree < 3.

   Answer: Since both integration, and evaluation are linear operations, it is enough to ensure that the weights are chosen to make the formula exact for some basis of the space $X = \{p : p \text{ is a polynomial}, \deg(p) < 3\}$. For convenience, we choose the standard basis 1, t, and t^2. (1) requires that

   $$2 = \int_{-1}^{1} 1dt = m_1 + m_2 + m_3 \tag{2a}$$

   $$0 = \int_{-1}^{1} tdt = -m_1 a + m_3 a \tag{2b}$$

   $$\frac{2}{3} = \int_{-1}^{1} t^2 dt = m_1 a^2 + m_3 a^2 \tag{2c}$$

   For (2b) to be satisfied, we must have that $m_1 = m_3$. (2c), then gives that $m_1 = m_3 = 1/(3a^2)$. Plugging this into (2a), then gives us $m_2 = 2 - 2/(3a^2)$. All together, we see that

   $$m_1 = \frac{1}{3a^2}, m_2 = 2 - \frac{2}{3a^2}, \text{ and } m_3 = \frac{1}{3a^2}.$$

(b) Show that for $a > \sqrt{1/3}$, all weights are positive.

Answer: Clearly m_1 and m_3 are positive for any value of $a \neq 0$. For m_2, note that

$$a > \sqrt{1/3} \Rightarrow \frac{1}{a^2} < 3 \Rightarrow \frac{2}{3a^2} < 2 \Rightarrow 2 - \frac{2}{3a^2} > 0,$$

so $m_2 > 0$, and all weights are positive.

(c) Show that for $a > \sqrt{3/5}$ (1) holds for all polynomials of degree < 6.

Answer: As in part a), we know that it is enough to check that (1) holds for the standard basis $1, t, \dots, t^5$. If we choose

$$m_1 = \frac{5}{9}, m_2 = 2 - \frac{10}{9} = \frac{8}{9}, \text{ and } m_3 = \frac{5}{9},$$

then part a), gives that (1) holds for $p(t) = 1, t, t^2$. We now check the others

$$-\frac{5a^3}{9} + 0 + \frac{5a^3}{9} = 0 = \int_{-1}^{1} t^3 dt \quad \checkmark$$
$$\frac{5a^4}{9} + \frac{5a^4}{9} = \frac{10}{9}\left(\frac{3}{5}\right)^2 = \frac{2}{5} = \int_{-1}^{1} t^4 dt \quad \checkmark$$
$$-\frac{5a^5}{9} + 0 + \frac{5a^5}{9} = 0 = \int_{-1}^{1} t^5 dt. \quad \checkmark$$

We see that (1) holds for all basis functions, and so we may conclude that it holds for all polynomials of degree < 6.

It is good to note that, by moving to non-equispaced sampling points, we were able to construct a far more accurate rule for numerical integration. The ideas here can be fairly easily extended to hold for different domains with an arbitrary number of sample points. For those that are interested, this method is called Guassian quadrature.

3. Let $\mathcal{P}_2$ be the linear space of all polynomials

$$p(x) = a_0 + a_1 x + a_2 x^2,$$

with real coefficients and degree ≤ 2. Let ξ_1, ξ_2, ξ_3 be three distinct real numbers, and then define

$$l_i = p(\xi_i), \ i = 1, 2, 3 \tag{3}$$

(a) Show that l_1, l_2, l_3 is a basis for $\mathcal{P}_2'$.

Answer: First, we note that the l_i's are evaluation operators, which are clearly linear functionals. Second, we note that we have three linear functionals, and by Theorem 2 in Chapter 2 of Lax

$$\dim(\mathcal{P}_2') = \dim(\mathcal{P}_2) = 3,$$

so l_1, l_2, l_3 form a basis iff they are linearly independent. To see that they are linearly independent, suppose that

$$b_1 l_1 + b_2 l_2 + b_3 l_3 = 0, \ b_1, b_2, b_3 \in \mathbb{R}$$

If we take this functionals and operate on $p_i = \prod_{j\neq i}(x-\xi_j)$, then we find that

$$0 = (b_1l_1 + b_2l_2 + b_3l_3)(p_i) = b_i \prod_{j\neq i}(\xi_i - \xi_j),\ i = 1, 2, 3.$$

Since the ξ_i's are distinct, this gives that $b_i = 0$ for all i, and the l_i's are linearly independent.

(b) i. Suppose that $\{e_1, \ldots, e_n\}$ form a basis for a vector space V. Show there exists linear functions $\{l_1, \ldots, l_n\}$ in V' defined by

$$l_i(e_j) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{4}$$

Show that $\{l_1, \ldots, l_n\}$ is a basis for V': it is called the dual basis.

Answer: Since $\{e_1, \ldots, e_n\}$ is a basis for V, we know that for any $v \in V$, we may write

$$v = \sum_{i=1}^{n} a_i e_i.$$

We will then define the l_i's on general v's by

$$l_i(v) = a_i. \tag{5}$$

Clearly this satisfies the required definition of l_i. To see that it is a linear functional, suppose

$$v = \sum_{i=1}^{n} a_i e_i, u = \sum_{i=1}^{n} b_i e_i \in V$$

then

$$l_i(\alpha v + \beta u) = l_i\left(\sum_{j=1}^{n}(\alpha a_j + \beta b_j)e_j\right) = \alpha a_i + \beta b_i = \alpha l_i(v) + \beta l_i(u),$$

as required. To see that $\{l_1, \ldots, l_n\}$ is a basis for V', it is enough to show that they are linearly independent because $\dim(V') = n = |\{l_1, \ldots, l_n\}|$. To see the linear independence, suppose that

$$\sum_{i=1}^{n} a_i l_i = 0.$$

If we evaluate this equation on e_j, we find that

$$0 = \sum_{i=1}^{n} a_i l_i(e_j) = \sum_{i=1}^{n} a_i \delta_{i,j} = a_j,\ j = 1, \ldots, n.$$

Thus all the a_i's are zero. Thus the l_i's are linearly independent and form a basis for V'.

ii. Find the basis $\{e_1, e_2, e_3\}$ in $\mathcal{P}_2$ for which the l_1, l_2, l_3 defined in (3) is the dual basis in $\mathcal{P}_2'$.

Answer: If we define

$$e_i(x) = \frac{\prod_{j\neq i}(x - \xi_j)}{\prod_{j\neq i}(\xi_i - \xi_j)},\ i = 1, 2, 3, \tag{6}$$

then we see that $l_i(e_j) = \delta_{i,j}$, as required. $\{e_1, e_2, e_3\}$ is clearly a basis because it has the required number of elements and is linearly independent for the same reason that the l_i's are.

4. Let W_1 and W_2 be two subspaces of a vector space V such that $H = W_1 \cup W_2$ is also a subspace of V. Show that one of the subspaces W_i is contained in the other.

Answer: Suppose $\exists w_1 \in W_1$ such that $w_1 \notin W_2$. Then, for all $w_2 \in W_2$, we have that

$$w_1 + w_2 \in H = W_1 \cup W_2,$$

because H is a subspace. Thus either $w_1 + w_2 \in W_1$ or W_2. If it is in W_2, we have that

$$w_1 = (w_1 + w_2) + (-w_2) \in W_2$$

because W_2 is a subspace, which is a contradiction. Thus it is in W_1 and we know that

$$w_2 = (w_1 + w_2) + (-w_1) \in W_1.$$

This shows that if $\exists w_1 \in W_1$ but not in W_2, then $w_2 \in W_1$ $\forall w_2 \in W_2$. Writing that another way, we have that $W_1 \not\subset W_2$ implies that $W_2 \subset W_1$. Since the labelling of

5. (a) Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.

Answer: If $Y \subset \mathbb{R}^1$ is a subspace of $\mathbb{R}^1$, then $\dim(Y) \leq \dim(\mathbb{R}^1) = 1$. If $\dim(Y) = 0$, then it is not possible to find any set of linearly independent vectors, so $Y = \{0\}$. If $\dim(Y) = 1$, then $Y = \mathbb{R}^1$, because any basis of Y must span the whole space.

(b) Prove that the only subspaces of $\mathbb{R}^2$ are $\mathbb{R}^2$, the zero subspace, or a scalar multiples of some fixed vector.

Answer: If $Y \subset \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$, then $\dim(Y) \leq \dim(\mathbb{R}^2) = 2$. If $\dim(Y) = 0$, then again Y is the zero subspace. If $\dim(Y) = \dim(R^2) = 2$, then again Y must be the whole space (R^2). Finally, if $\dim(Y) = 1$, then by the definition of dimension, $Y = \text{span}\{v\}$, for some vector $v \in Y$. The only linear combinations of a single vector, are scalar multiples of the vector, so it must be that $Y = \{cv : c \in \mathbb{R}\}$.

(c) Describe all the subspaces of $\mathbb{R}^3$.

Answer: If $Y \subset \mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, then $\dim(Y) \leq \dim(\mathbb{R}^3) = 3$. If $\dim(Y) = 0$, then Y is the zero subspace. If $\dim(Y) = \dim(\mathbb{R}^3) = 3$, then $Y = \mathbb{R}^3$. If $\dim(Y) = 1$, then again $Y = \{cv : c \in \mathbb{R}\}$, for some $v \in \mathbb{R}^3$. Finally, if $\dim(Y) = 2$, then $Y = \text{span}\{u, v\}$, for some $u, v \in \mathbb{R}^3$. The set of linear combinations of two vectors forms a plane containing the origin $Y = \{cu + dv : c, d \in \mathbb{R}\}$. Thus the subspaces of $\mathbb{R}^3$ are of the form

$$\{0\}, \{cv : c \in \mathbb{R}\}, \{cu + dv : c, d \in \mathbb{R}\}, \text{ and } \mathbb{R}^4.$$

6. Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove that the vector space is not finite dimensional.

   Answer: There are two main ways to answer this question:

   Cardinality approach: If a vector space V over the field of rationals is finite dimensional, then it has a basis $\{e_1, \dots, e_n\}$ and may be written

   $$V = \{a_1 e_1 + \cdots + a_n e_n : a_1, \dots, a_n \in \mathbb{Q}\}.$$

   We may then uniquely identify any element $v \in V$ with vector in $\mathbb{Q}^n$, and so we see that the cardinality of V is the same as $\mathbb{Q}^n$, which is a countable set. Since $\mathbb{R}$ is an uncountable set, it cannot be a finite dimensional vector space over $\mathbb{Q}$.

   Basis construction approach: Consider the set of vectors $\{1, \pi, \dots, \pi^n\}$. If the rational numbers $a_0, \dots, a_n$ are such that

   $$a_0 + a_1 \pi + \cdots + a_n \pi^n = 0,$$

   then we know that π is a root of the polynomial

   $$a_0 + a_1 x + \cdots + a_n x^n.$$

   Since π is known to be a transcendental number, and this is a polynomial with rational coefficients, it must be that $a_0 = \cdots = a_n$, and so $\{1, \pi, \dots, \pi^n\}$ is a set of linearly independent vectors for any $n \in \mathbb{N}$. Since it is possible to find a set of linearly independent vectors of any size, it must be that V is infinite dimensional.