1. Let $\sum_{j=1}^{n} t_{ij}x_j = u_i$, $i = 1, \dots, m$ be an overdetermined set of linear equations (i.e. $m > n$). Take the case that in spite of this, the systems of equations has a unique solution for some given, and fixed $u_1, \dots, u_m$. Show that it is possible to select a subset of n of these equations which uniquely determines that solution.

   Answer: We may express this systems of equations in matrix form as

   $$Tx = u,$$

   where $T \in \mathbb{R}^{m\times n}$. We know that for some fixed u, $\exists!\, x$ such that $Tx = u$. This implies that $Ty = v$ has a unique solution for any $v \in \mathbb{R}^m$ because if $Ty_1 = Ty_2 = v$, then

   $$T(x + y_1 - y_2) = Tx + Ty_1 - Ty_2 = u.$$

   Thus, by the uniqueness of x, we have that $y_1 = y_2$. We have thus shown that T is injective and $N_T = \{0\}$. By the rank nullity theorem, we have that

   $$\text{rank}(T) = n - \text{nullity}(T) = n.$$

   Since the rank of T is n, we know that there exists n linearly independent rows of T $\{t_{i_1}, \dots, t_{i_n}\}$. If we let $\tilde{T}$ be the matrix with rows t_{i_j}, then we know that $\text{rank}(\tilde{T}) = n$, so T is invertible by the rank nullity theorem, and the system

   $$\tilde{T}\tilde{x} = \begin{bmatrix} u_{i_1} \\ \vdots \\ u_{i_n} \end{bmatrix} \tag{1}$$

   uniquely determines $\tilde{x}$. This $\tilde{x}$ is the solution of the original system of equations x because (1) is contained in the original system of equations and $\tilde{x}$ is unique. Thus it is possible to choose a subset of n of the original equations (1), which uniquely determines x.

2. Let S be rotation around the x_1 axis by $\pi/2$ in $\mathbb{R}^3$ and let T be rotation around the x_2 axis by $\pi/2$ in $\mathbb{R}^3$. Show that $ST \neq TS$.

   Answer: To do this, it is enough to show that for some $x \in R^3$, $ST(x) \neq TS(x)$. In particular, we let

   $$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

   Then

   $$T(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } S(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

   and

   $$S(T(x)) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \text{ and } T(S(x)) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   and so $ST(x) \neq TS(x)$.

3. Let $S : V \to W$, $T : U \to V$, and $R : U \to V$. Show that

   (a) $(ST)' = T'S'$.

   Answer: For any $l \in W'$ and $x \in U$, we have that

   $$\big((ST)'l, x\big) = (l, STx) = \big(S'l, Tx\big) = \big(T'S'l, x\big)\,.$$

   Since this holds $\forall l$, $\forall x$, we have that $(ST)' = T'S'$.

   (b) $(T + R)' = T' + R'$.

   Answer: For any $l \in V'$ and $x \in U$, we have that

   $$\big((T+R)'l, x\big) = (l, (T+R)x) = (l, Tx) + (l, Rx) = \big(T'l, x\big) + \big(R'l, x\big) = \big((T'+R')l, x\big)\,.$$

   since this holds $\forall l$, $\forall x$, we have that $(T + R)' = T' + R'$

   (c) $(T^{-1})' = (T')^{-1}$, if either exists.

   Answer: Suppose that T^{-1} exists. Then we know

   $$TT^{-1} = \mathrm{id}$$
   $$\Rightarrow (T^{-1})'T' = \mathrm{id}',$$

   by part a). Thus, $(T^{-1})'$ is a left inverse for T'. This is enough to show that $(T^{-1})' = (T')^{-1}$, but we could also repeat the above using that T^{-1} is a left inverse of T to show that $(T^{-1})'$ is also a right inverse for T'.
   If $(T')^{-1}$ exists, then by the above $((T')^{-1})' = (T'')^{-1}$. Since $T = T''$, we are done.
   Note: It is possible to prove that if one inverse exists then the other must exist before finding our formula by proving that the dual map is injective and surjective. These are useful arguments, and we will need them later, but it is important to observe that it is enough to show one. If you have shown that a map S is surjective, then you have shown that it has maximum rank, which, by the rank nullity theorem, tells you that it has zero nullity, i.e. it is injective.
   Similarly, if you show that S is injective (nullity zero), then the rank nullity theorem tells you that it is surjective. Thus, in finite dimensions, one need not check both.

4. Describe explicitly a linear transform from $\mathbb{R}^3$ to $\mathbb{R}^3$ which has it's range subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.

   Answer: Any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$may be described by a matrix. For a matrix, we know that it's range is the span of it's columns, so we need only construct a matrix A whose columns space the desired space. For example

   $$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 2 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ -1 & 2 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

   It is also possible to describe T without a matrix. For example the first matrix corresponds to the transformation

   $$T(x, y, z) = (x + y, 2y, -x + 2y).$$

5. Let V by an n-dimensional vector space over a field K and let T be a linear transformation from V into Vsuch that the range and null space of T are identical. Prove that n is even and give an example of such a linear transformation T.

   Answer: We are told that the range and null space of T are identical, so $\dim(R_T) = \dim(N_T)$. By the rank nullity theorem,

$$n = \dim V = \dim R_T + \dim N_T = 2 \dim R_T.$$

   Thus $n = 2\dim(R_T)$ and so even.

   An example of such a linear transformation is $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. To see this, we compute

$$Tx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}.$$

   Clearly $Tx = 0$ iff $x_2 = 0$ ($N_T = \{(x_1, 0) : x_1 \in \mathbb{R}\}$) and x is in the range of T iff $x_2 = 0$ ($R_T = \{(x_1, 0) : x_1 \in \mathbb{R}\} = N_T$).

6. Let V be a vector space and $T \in \mathcal{L}(V, V)$. Prove that the following statements are equivalent.

   (a) $R_T \cap N_T = \{0\}$

   (b) If $T(Tx) = 0$, then $Tx = 0$.

   Proof that a) ⇒ b). a) states that $R_T \cap N_T = \{0\}$. Then let x be such that $T(Tx) = 0$. This is the statement that $Tx \in N_T$. It is also clear that $Tx \in R_T$, and so by our assumption $Tx = 0$. □

   Proof that b) ⇒ a). Suppose that $y \in R_T \cap N_T$. Then, since $y \in R_T$ $\exists$ $x \in V$ such that $Tx = y$. Since $y \in N_T$, we have that $T(Tx) = 0$. By assumption this gives that $y = Tx = 0$. Thus for any $y \in R_T \cap N_T$, $y = 0$. i.e. $R_T \cap N_T = \{0\}$. □

7. Let $T \in \mathcal{L}(X, X)$ for some vector space X, $\dim X < \infty$. Suppose that $\dim R_T = \dim R_{T^2}$. Show that $R_T \cap N_T = \{0\}$.

   Answer: By the rank nullity theorem, we have that

$$\dim N_T = n - \dim R_T = n - \dim R_{T^2} = \dim N_{T^2}.$$

   Clearly $N_T \subseteq N_{T^2}$, (You can see this by noting that if $Tx = 0$, then $T^2x = T0 = 0$.) so it must be that $N_T = N_{T^2}$. i.e. if $T(Tx) = 0$, then $Tx = 0$. Thus question 6 tells us that $R_T \cap N_T = \{0\}$.