1. Show by Gaussian elimination that the only left null vectors of

$$M = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 3 \\ 3 & 4 & 6 & 2 \end{pmatrix}$$

are of the multiples of $l = \begin{pmatrix} 1 & -2 & -1 & 1 \end{pmatrix}$. Then use the fact that for a linear map T, $R_T^\perp = N_{T'}$ to conclude that the condition $0 = u_4 - u_3 - 2u_2 + u_1$ is necessary and sufficient to solve the system $Mx = u$.

Answer: To find the left null vectors of M, we perform Gaussian elimination on M^T.

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 1 & 4 \\ 2 & 3 & 2 & 6 \\ 3 & 1 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & -3 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We may now use back-substitution to find l. First we see $l_4 = t$ is a free variable. The next row tells us that $l_3 = -l_4 = -t$. Row 2 says $l_2 = l_3 - l_4 = 2t$. Row 1 tells us that $l_1 = -l_2 - 2l_3 - 3l_4 = t$. Altogether we have that $l = t(1, -2, -1, 1)$, so $N_{T'} = \text{span}\{(1, -2, -1, 1)\}$. We know that $R_T^\perp = N_{T'} = \text{span}\{(1, -2, -1, 1)\}$, so

$$\exists\, x \text{ s.t. } Mx = u \Leftrightarrow u \in R_T \Leftrightarrow l(u) = 0\ \forall l \in R_T^\perp$$
$$\Leftrightarrow t(u_1 - 2u_2 + -u_3 + u_4) = 0\ \forall t \Leftrightarrow u_1 - 2u_2 + -u_3 + u_4 = 0.$$

i.e. $Mx = u$ is solvable if and only if $u_1 - 2u_2 + -u_3 + u_4 = 0$.

2. Suppose $T \in \mathcal{L}(X)$, $\dim X = n$ and let $B : X \to \mathbb{R}^n$ be an isomorphism such that $B\alpha_i = e_i$, $i = 1, \dots, n$ for some basis $\mathcal{B} = \{\alpha_1, \dots \alpha_n\}$ of X. Let $M = BTB^{-1} \in \mathcal{L}(\mathbb{R}^n)$ and let $M_{ij} = (Me_j)_i$ be the matrix associated with M as in Theorem 1 pg 32 (Lax). Show that $T\alpha_j = \sum_{i=1}^n M_{ij}\alpha_i$, $i = 1, \dots, n$. Thus M_{ij} is the matrix for T in the basis $\mathcal{B}$.

Answer: We use linearity of B and our definitions to see that

$$T\alpha_j = B^{-1}MB\alpha_j = B^{-1}Me_j = B^{-1}\sum_{i=1}^n (Me_j)_i e_i = \sum_{i=1}^n M_{ij}\alpha_i.$$

This is the definition of M being the matrix representation of T in the basis $\mathcal{B}$.

3. Let S be a linear operator in R^2 such that $S^2 = S$ (i.e. S is a projection). Show that either $S = 0$ or $S = I$ **or** $S\alpha_j = \sum_{i=1}^2 A_{ij}\alpha_i$ $j = 1, 2$ for some basis (α_1, α_2) for $\mathbb{R}^2$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Note: I understand this question was hard to read on the sheet, but the question only makes sense if the word after "$S = I$" is "or", and so the homework was marked accordingly.

Answer: Since $S \in \mathcal{L}(\mathbb{R}^2)$, we know that $\text{rank}(S) = 0, 1,$ or 2. If $\text{rank}(S) = 0$, then $R_S = \{0\}$, and so $S = 0$. If $\text{rank}(S) = 2$, then $R_S = \mathbb{R}^2$. Thus $\forall y \in \mathbb{R}^2$ there is an x such that $Sx = y$. The fact that $S^2 = S$ tells us that $Sy = S^2x = Sx = y$, and so $Sy = y$ for all $y \in \mathbb{R}^2$ and $S = I$.

If $\operatorname{rank}(S) = 1$, then $\dim N_S = \dim R_S = 1$ and so there exists $\alpha_1 \neq 0$ such that $R_S = \operatorname{span}\{\alpha_1\}$ and $\alpha_2 \neq 0$ such that $N_S = \operatorname{span}\{\alpha_2\}$. $S^2 = S$ tells us that if $S^2 x = 0$ then $Sx = 0$. By question 6 of homework 2, we have that $R_S \cap N_S = \{0\}$. Thus $\{\alpha_1, \alpha_2\}$ is a linearly independent set of 2 vectors, and so a basis.

To find the matrix of S in the basis $\{\alpha_1, \alpha_2\}$, note that $\alpha_1 \in \mathbb{R}_S$ implies that there is a y so that $Sy = \alpha_1$, then $S\alpha_1 = S^2 y = Sy = \alpha_1$. Now write $x = c_1\alpha_1 + c_2\alpha_2$, and so

$$Sx = c_1 S(\alpha_1) + c_2 S(\alpha_2) = c_1\alpha_1.$$

Thus the matrix for S is the required A.

4. Let X be an n-dimensional vector space over a field K, and let $\mathcal{B}\{\alpha_1, \dots, \alpha_n\}$ be a basis for X.

   (a) Show that there is a unique linear operator T on X such that $T\alpha_j = \alpha_{j+1}$, $j = 1, \dots, n-1$, and $T\alpha_n = 0$. What is the matrix A of T in the basis $\mathcal{B}$. i.e. $T\alpha_i = \sum_{i=1}^{n} A_{ij}\alpha_i$, $i = 1, \dots, n$

   Answer:
   If $x = \sum_{i=1}^{n} c_i\alpha_i$, we will define $Tx = \sum_{i=1}^{n-1} c_i\alpha_{i+1}$. If $y = \sum_{i=1}^{n} d_i\alpha_i$, then

$$T(ax + by) = \sum_{i=1}^{n-1}(ac_i + bd_i)\alpha_{i+1} = a\sum_{i=1}^{n-1} c_i\alpha_{i+1} + b\sum_{i=1}^{n-1} d_i\alpha_{i+1} = aTx + bTy,$$

   so T is a linear operator. Also $T\alpha_j = \alpha_{j+1}$, $j = 1, \dots, n-1$, and $T\alpha_n = 0$.
   Any linear transformation is uniquely determined by it's action on a basis, so T is unique. To see this explicitly, suppose S satisfies our desired property, then

$$T(\alpha_i) = \alpha_{i+1} = S(\alpha_i), i = 1, \dots, n-1, \text{ and } T(\alpha_n) = 0 = S(\alpha_n)$$

   and so $T(x) = S(x)$ for all $x \in X$, i.e. $T = S$, so T is unique.
   Clearly $T\alpha_j = \sum_{i=1}^{n} \delta_{i,j+1}\alpha_i$, and so the matrix is

$$A = \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}$$

   (b) Prove that $T^n = 0$ and $T^{n-1} \neq 0$.

   Answer: Repeated applications of T on the basis vectors clearly gives

$$T^k\alpha_i = T^{k-1}\alpha_{i+1} = \cdots = \alpha_{i+k},$$

   where $\alpha_j = 0$ if $j > n$. We thus see that $T^{n-1}\alpha_1 = \alpha_n \neq 0$, and so $T^{n-1} \neq 0$. We also see that $T^n\alpha_i = 0$ for all i. Thus T^n send all basis vectors to 0, and so $T^n = 0$.

   (c) Let S be any linear operator on X such that $S^n = 0$, but $S^{n-1} \neq 0$. Prove that there is a basis $\mathcal{B}'$ for X such that the matrix for S in the basis $\mathcal{B}'$ is the matrix A from part a).

Answer: Since $S^{n-1} \neq 0$, there exists an α_1 such that $S^{n-1}\alpha_1 \neq 0$. If we let $\alpha_j = S^{j-1}\alpha_1$, I claim that $\mathcal{B}' = \{\alpha_1, \dots, \alpha_n\}$ is a basis for X. Clearly it has the right number of elements, so we need only check linear independence.
Suppose $c_1, \dots, c_n$ are such that

$$c_1\alpha_1 + \cdots c_n S^{n-1}\alpha_1 = 0$$

then applying S^{n-1} to both sides gives

$$c_1 S^{n-1}\alpha_1 + S^n(\alpha_1 + S\alpha_1 + \cdots + S^{n-2}\alpha_1) = 0.$$

By the definition of α_1, and the fact that $S^n = 0$, this tells us that $c_1 = 0$. We repeat this process by multiplying by S^{n-j} to show that all of the c_j's are zero and so $\mathcal{B}'$ is a set of n linearly independent vectors, and so a basis for X.
This basis also clearly satisfies the property that

$$S\alpha_j = S^j\alpha_j = \alpha_{j+1}, j = 1, \dots, n-1, \text{ and } S\alpha_n = S^n\alpha_1 = 0\alpha_1 = 0.$$

Thus S satisfies the same properties that defined T in part a), and so has the same matrix representation

(d) Prove that M and N are $n \times n$ matrices over K such that $M^n = N^n = 0$ but $M^{n-1} \neq 0$ and $N^{n-1} \neq 0$, then M and N are similar.

Answer: By part c), there exists bases $\mathcal{B}_1$ and $\mathcal{B}_2$ such that writing M and N in those respective bases gives the same matrix representation A. Representing these change of basis operations by the matrices P_1 and P_2, we see that

$$P_1 M P_1^{-1} = A = P_2 N P_2^{-1}$$
$$\Rightarrow M = P_1^{-1} P_2 N P_2^{-1} P_1 = (P_1^{-1}P_2) N (P_1^{-1}P_2)^{-1},$$

and so M and N are similar.

5. Let W_1 and W_2 be subspaces of a finite-dimensional vector space X

(a) Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.

Answer: Let $l \in (W_1 + W_2)^\perp$, then for all $w_1 \in W_1$ and $w_2 \in W_2$, $l(w_1 + w_2) = 0$. In particular, choosing $w_2 = 0$ (allowed since W_2 is a subspace) gives that $l(w_1) = 0$ for all $w_1 \in W_1$, so $l \in W_1^\perp$. Choosing $w_1 = 0$ similarly shows that $l \in W_2^\perp$. Thus $l \in W_1^\perp \cap W_2^\perp$. Since this is true for all $l \in (W_1 + W_2)^\perp$, we have shown that $(W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp$.
Now suppose $l \in W_1^\perp \cap W_2^\perp$. Then for all $w_1 \in W_1$ and $w_2 \in W_2$, $l(w_1) + l(w_2) = 0$. Using linearity, we see that

$$l(w_1 + w_2) = l(w_1) + l(w_2) = 0,$$

so $l \in (W_1 + W_2)^\perp$. Since this is true for all $l \in W_1^\perp \cap W_2^\perp$, we have that $W_1^\perp \cap W_2^\perp \subset (W_1 + W_2)^\perp$.
Since we have shown containment in both directions, we have the desired equality.

(b) Prove that $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$.

Answer: If we let $Z_1 = W_1^\perp$ and $Z_2 = W_2^\perp$, then part a) tells us that

$$(Z_1 + Z_2)^\perp = Z_1^\perp \cap Z_2^\perp$$
$$\Rightarrow (W_1^\perp + W_2^\perp)^\perp = (W_1^\perp)^\perp \cap (W_2^\perp)^\perp.$$

Using the fact that for any subspace Y, $(Y^\perp)^\perp = Y$, we have that

$$(W_1^\perp + W_2^\perp)^\perp = W_1 \cap W_2$$
$$\Rightarrow W_1^\perp + W_2^\perp = (W_1 \cap W_2)^\perp.$$