1. Show that if A is a square matrix, then $\det A = \det A^T$, where A^T is the transpose of A.

   Answer: First we note that if σ is a permutation, then $\text{sign}(\sigma^{-1}) = \text{sign}(\sigma)$ because

$$1 = \text{sign}(\text{id}) = \text{sign}(\sigma^{-1} \circ \sigma) = \text{sign}(\sigma^{-1})\text{sign}(\sigma).$$

   Next, we shall use Leibniz's formula for the derivative

$$\det A = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i}$$

   If we let $j = \sigma^{-1}(i)$ and use the fact that we may rearrange the order of multiplication, we find that

$$\begin{aligned}\det A &= \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j=1}^{n} a_{\sigma(\sigma^{-1}(j))\sigma^{-1}(j)} \\ &= \sum_{\sigma \in S_n} \text{sign}(\sigma^{-1}) \prod_{j=1}^{n} a_{j\sigma^{-1}(j)} \\ &= \sum_{\tau \in S_n} \text{sign}(\tau) \prod_{j=1}^{n} a_{j\tau(j)} \text{ by letting } \tau = \sigma^{-1} \\ &= \sum_{\tau \in S_n} \text{sign}(\tau) \prod_{j=1}^{n} (A^T)_{\tau(j)j} = \det A^T.\end{aligned}$$

2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ be a matrix over a field K and (c_1, c_2, c_3) be a vector in K^3 be defined by

$$c_1 = \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}, c_2 = \det \begin{pmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{pmatrix}, \text{ and } c_3 = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

   Show that

   (a) $\text{rank}(A) = 2$ if and only if $(c_1, c_2, c_3) \neq (0, 0, 0)$.

   Answer: First suppose that $\text{rank}(A) = 2$, then A has at least one pair of linearly independent columns. Without loss of generality, suppose that it is the first two. Then

$$c_3 = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0.$$

   Now suppose that $(c_1, c_2, c_3) \neq (0, 0, 0)$. then at least one of the c_i's is non zero. By the definition of the c_i's this says that A has at least 2 linearly independent columns, which implies that $\text{rank}(A) \geq 2$. Since A is a 3×2 matrix, it has rank at most 2, and so $\text{rank}(A) = 2$.

   (b) If $\text{rank}(A) = 2$, then (c_1, c_2, c_3) is a basis for the solution space of the equation $Ax = 0$.

Answer: By the rank nullity theorem nullity$(A) = 3 - \text{rank}(A) = 1$, so the solution space of $Ax = 0$ is one dimensional. Since part a) tells us that $(c_1, c_2, c_3) \neq 0$, it is enough to show that it is in the solution space.

$$A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} a_{12}a_{23} - a_{13}a_{22} \\ a_{13}a_{21} - a_{11}a_{23} \\ a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}a_{12}a_{23} - a_{11}a_{13}a_{22} + a_{12}a_{13}a_{21} - a_{12}a_{11}a_{23} + a_{13}a_{11}a_{22} - a_{13}a_{12}a_{21} \\ a_{21}a_{12}a_{23} - a_{21}a_{13}a_{22} + a_{22}a_{13}a_{21} - a_{22}a_{11}a_{23} + a_{23}a_{11}a_{22} - a_{23}a_{12}a_{21} \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3. Prove that for an real numbers $x_1, \ldots, x_n$,

$$V = \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \leq j < i \leq n} (x_i - x_j).$$

This determinant is called the Vandermonde determinant.

Answer: There are two usual proofs of this determinant. The most obvious and straightforward is direct Gaussian elimination, but that is fairly messy. I outline the other approach below.

First we note that the the Laplace formula applied to the jth row tells us that

$$V = (-1)^j \det \begin{pmatrix} \ddots \end{pmatrix} + (-1)^{j+1} x_j \begin{pmatrix} \ddots \end{pmatrix} + \cdots + (-1)^{j+n-1} x_j^{n-1} \begin{pmatrix} \ddots \end{pmatrix},$$

as each of the remaining matrices is independent of x_j, V is an $(n-1)$ degree polynomial in x_j for every x_j.

Next, we note that if $x_i = x_j$ for any $i \neq j$, then two rows of our matrix are identical and so $V = 0$. Thus we may write

$$V = Q(x_1, \ldots, x_n) \prod_{1 \leq j < i \leq n} (x_i - x_j).$$

Since the product on the right is an $n-1$th degree polynomial of x_j for every j, so Q cannot depend on x_j for any j, and so Q is constant. To determine the value of this constant, we use the Leibniz formula for the determinant to see that the the term $x_2 x_3^2 \cdots x_n^{n-1}$ only appears for $\sigma =$id, and so will have coefficient 1. This monomial is also only obtained by taking the first term in every factor of our product, and so we have that $x_2 x_3^2 \cdots x_n^{n-1} = Q x_2 x_3^2 \cdots x_n^{n-1}$, and so $Q = 1$ and

$$V = \prod_{1 \leq j < i \leq n} (x_i - x_j).$$

4. An $n \times n$ matrix is triangular if $A_{ij} = 0$ whenever $i > j$ (upper triangular) or $i < j$ (lower triangular). Show that $\det A = A_{11} \cdots A_{nn}$ if A is triangular.

Answer: We prove this by induction. If $n = 1$, then clearly $\det A = A_{11}$, and so the claim holds.

Now suppose the result holds for $n-1$ and suppose that A is an $n \times n$ upper triangular matrix. Then, by the Laplace formula for the determinant

$$\det A = \det \begin{pmatrix} A_{11} & A_{12} & \cdots & \\ & A_{22} & A_{23} & \\ & & \ddots & \\ 0 & & & A_{nn} \end{pmatrix} = A_{11} \det \begin{pmatrix} A_{22} & A_{23} & \cdots & \\ & A_{33} & A_{34} & \\ & & \ddots & \\ 0 & & & A_{nn} \end{pmatrix} + 0 \det \left(\ddots \right) + \cdots + 0 \det \left(\ddots \right).$$

By the induction hypothesis, we then have that

$$\det A = \det \begin{pmatrix} A_{11} & A_{12} & \cdots & \\ & A_{22} & A_{23} & \\ & & \ddots & \\ 0 & & & A_{nn} \end{pmatrix} = A_{11} A_{22} \cdots A_{nn},$$

and so the claim holds for upper triangular matrices. If A is lower triangular then A^T is upper triangular and has the same diagonal as A and so the result holds. Thus the result holds for any square triangular matrix of any size.

5. If A and B are $n \times n$ matrices then $\det AB = \det BA$.

   (a) If A is $n \times q$ and B is $q \times n$, with $q \geq n$, prove the Cauchy-Binet formula

$$\det AB = \sum_{1 \leq i_1 < \cdots < i_n \leq q} \det \begin{pmatrix} a_{1i_1} & \cdots & a_{1i_n} \\ \vdots & & \vdots \\ a_{ni_1} & \cdots & a_{ni_n} \end{pmatrix} \det \begin{pmatrix} b_{i_1 1} & \cdots & b_{i_n n} \\ \vdots & & \vdots \\ b_{i_1 1} & \cdots & b_{i_n n} \end{pmatrix},$$

   where $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq q}}$ and $B = (b_{ij})_{\substack{1 \leq i \leq q \\ 1 \leq j \leq n}}$.

   Answer: By definition $(AB)_{ij} = \sum_k a_{ik} b_{kj}$. Thus, using the multilinearity of the determinant, we see that

$$\begin{aligned} \det AB &= \det \begin{pmatrix} \sum_i a_{1i} b_{i1} & \cdots & \sum_i a_{1i} b_{in} \\ \vdots & & \vdots \\ \sum_k a_{ni} b_{i1} & \cdots & \sum_i a_{ni} b_{in} \end{pmatrix} \\ &= \sum_{i_1, \ldots, i_n} \det \begin{pmatrix} a_{1i_1} b_{i_1 1} & \cdots & a_{1i_n} b_{i_n n} \\ \vdots & & \vdots \\ a_{ni_1} b_{i_1 1} & \cdots & a_{ni_n} b_{i_n n} \end{pmatrix} \\ &= \sum_{i_1, \ldots, i_n} b_{i_1 1} \cdots b_{i_n n} \det \begin{pmatrix} a_{1i_1} & \cdots & a_{1i_n} \\ \vdots & & \vdots \\ a_{ni_1} & \cdots & a_{ni_n} \end{pmatrix}. \end{aligned}$$

   Clearly, if $i_k = i_j$ for some $j \neq k$, two columns of the matrix in the determinant will be identical, and so the determinant will be unique. Thus sum may be reduced to a sum over sets where all of the i_k's are distinct. We may then split this into a sum over increasing sets of indices and a sum over permuations of those indices

$$\begin{aligned}
\det AB &= \sum_{1\le i_1<\cdots<i_n\le q} \sum_{\sigma\in S_n} b_{i_{\sigma(1)}1}\cdots b_{i_{\sigma(n)}n} \det\begin{pmatrix} a_{1i_{\sigma(1)}} & \cdots & a_{1i_{\sigma(n)}} \\ \vdots & & \vdots \\ a_{ni_{\sigma(1)}} & \cdots & a_{ni_{\sigma(n)}} \end{pmatrix} \\
&= \sum_{1\le i_1<\cdots<i_n\le q} \left(\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) b_{i_{\sigma(1)}1}\cdots b_{i_{\sigma(n)}n}\right) \det\begin{pmatrix} a_{1i_1} & \cdots & a_{1i_n} \\ \vdots & & \vdots \\ a_{ni_1} & \cdots & a_{ni_n} \end{pmatrix} \\
&= \sum_{1\le i_1<\cdots<i_n\le q} \det\begin{pmatrix} a_{1i_1} & \cdots & a_{1i_n} \\ \vdots & & \vdots \\ a_{ni_1} & \cdots & a_{ni_n} \end{pmatrix} \det\begin{pmatrix} b_{i_11} & \cdots & b_{i_nn} \\ \vdots & & \vdots \\ b_{i_11} & \cdots & b_{i_nn} \end{pmatrix}.
\end{aligned}$$

(b) Use (a) to evaluate $\det AB$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}.$$

Answer: Applying the Cauchy-Binet formula gives

$$\begin{aligned}
\det AB &= \sum_{1\le i_1<i_2\le 3} \det\begin{pmatrix} a_{1i_1} & a_{1i_2} \\ a_{2i_1} & a_{2i_2} \end{pmatrix} \det\begin{pmatrix} b_{i_11} & b_{i_12} \\ b_{i_21} & b_{i_22} \end{pmatrix} \\
&= \sum_{(i_1,i_2)\in\{(1,2),(1,3),(2,3)\}} (a_{1i_1}a_{2i_2} - a_{1i_2}a_{2i_1})(b_{i_11}b_{i_22} - b_{i_12}b_{i_21}) \\
&= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) + (a_{11}a_{23} - a_{13}a_{21})(b_{11}b_{32} - b_{12}b_{31}) \\
&\qquad + (a_{12}a_{23} - a_{13}a_{22})(b_{21}b_{32} - b_{22}b_{31}).
\end{aligned}$$

(c) If $q < n$ in (a), show that $\det AB = 0$.

Answer: There are two main ways to prove this. One is to prove that that AB is not invertible by a rank-nullity argument.

The other method is to note that the sum in the formula from part (a) is empty because there is no way to pick n distinct i_j's such that $1 \le i_j \le q$ for each j.

6. Let A be the $n \times n$ matrix

$$\begin{pmatrix} x & a & & \cdots & a \\ a & x & a & \cdots & a \\ \vdots & & \ddots & & \vdots \\ & & a & x & a \\ a & & \cdots & a & x \end{pmatrix}.$$

Compute $\det A$ as a function of x and a.

Answer: The most straightforward proof is by Gaussian elimination as follows:

$$
\begin{aligned}
\det A &= \det \begin{pmatrix} x & a & & \cdots & a \\ a & x & a & \cdots & a \\ \vdots & & \ddots & & \vdots \\ & & a & x & a \\ a & & \cdots & a & x \end{pmatrix} \\
&= \det \left(\begin{matrix} r_1 \\ r_2 - r_1 \\ \\ \\ r_n - r_1 \end{matrix} \begin{pmatrix} x & a & \cdots & a & a \\ a-x & x-a & & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & a-x & \cdots & x-a & 0 \\ 0 & 0 & \cdots & 0 & x-a \end{pmatrix} \right) \\
&= \det \left(\begin{matrix} \sum_{j=1}^n c_j & \sum_{j=2}^n c_j & & c_{n-1}+c_n & c_n \end{matrix} \\ \begin{pmatrix} x+(n-1)a & (n-1)a & \cdots & 2a & a \\ 0 & x-a & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & x-a & 0 \\ 0 & 0 & \cdots & 0 & x-a \end{pmatrix} \right) \\
&= (x-a)^{n-1}(x+(n-1)a), \text{ by question 4.}
\end{aligned}
$$

It is also possible to prove this by using the fact the $\det A$ will be a polynomial of degree $n-1$ in x and finding what that polynomial is, but it is slightly more convoluted than the method presented above.

7. Let A, B, C, and D be square matrices of the same size. If $AC = CA$, then show that $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB)$.

(Note that this is slightly different than the original question. Some experimentation will tell you that the original claim is not true under these assumptions, whereas this one claim is.)

Answer: First suppose that A is invertible. We start by noting that

$$
\begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} \begin{pmatrix} I_n & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
$$

Thus we have reduced the problem to finding the determinant of the two matrices on the left.

First, we use the Laplace formula to expand along the last column to see that

$$
\begin{aligned}
\det \begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} &= 1 \det \begin{pmatrix} A & 0 \\ (C_{ij})_{\substack{1\le i\le n-1 \\ 1\le j\le n-1}} & I_{n-1} \end{pmatrix} + 0 \det \left(\ddots \right) \\
&\ \ \vdots \\
&= \det A.
\end{aligned}
$$

The same argument applied to $\begin{pmatrix} I_n & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}$ gives that $\det \begin{pmatrix} I_n & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} = \det(D - CA^{-1}B)$. We thus have that

$$\begin{aligned} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det \begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} \det \begin{pmatrix} I_n & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} \\ &= \det A \det(D - CA^{-1}B) \\ &= \det(AD - ACA^{-1}B) \\ &= \det(AD - CB). \end{aligned}$$

Now we wish to prove the above formula for general A. To do this we shall apply the above result to $A + tI$ and take the limit as $t \to 0$.

We recall that A is invertible if and only $\det A \neq 0$. Thus to show that $A + tI_n$ is invertible it is enough to show that $\det(A + tI) \neq 0$ for small t. Because the determinant is a polynomial, we know $\det(A + tI)$ is a polynomial in t and so we know that whether or not $\det A = 0$, there is some punctured neighbourhood of 0, where $\det(A + tI) \neq 0$.

Also note that if $AC = CA$, then $(A + tI)C = AC + tC = CA + tC = C(A + tI)$, and so for t in our allowable punctured neighbourhood, the above result says that

$$\det \begin{pmatrix} (A + tI) & B \\ C & D \end{pmatrix} = \det((A + tI)D - CB).$$

If we again recall that the determinant of a matrix is polynomial in its elements, and so continuous in the elements of the matrix, we may take the limit of each side as $t \to 0$, giving

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$