1. Let X be the linear space of lower tirangular matrixes. Show that the space of symmetric matrices S is (isomorphic to) the dual space X' of X.

   Answer: It is possible to show that all finite dimensional vector spaces of the same dimension n are isomorphic through showing they are all isomorphic to $\mathbb{R}^n$. To apply it to this problem we may construct a basis $\{E_{i,j}\}$ for X (e.g. $(E_{i,j})_{k,l} = \delta_{k,i}\delta_{l,j}$) and then note that $\{E_{i,j} + E_{i,j}^T\}$ is a basis for S, and so they have the same dimension. I will not pursue this approach because the hint indicates that we are looking for an explicit, natural isomorphism between S and X'.

   We claim that this isomorphism is the map $l : S \to X'$, where $l_A(E) = \mathrm{tr}[AE]$. Clearly l_A is linear for any $A \in S$ by the linearity of the trace (so $l_A \in X'$ $\forall A \in S$), and l is a linear map by the linearity of the trace. We also know that $\dim S = \dim X = \dim X'$, so l is an isomorphism if and only if it is one-to-one.

   To see this, we note that if we let $E_{i,j} \in X$ be as defined above, then $l_A(E_{i,j}) = A_{i,j}$, so if $A, A' \in S$ are such that $l_A = l_{A'}$, then $A_{i,j} = A'_{i,j}$ for every $j \leq i$. By the fact that A and A' are symmetric, we have that $A = A'$, thus l is one-to-one, and so we have found our isomorphism.

   Note: This is the natural isomorphism in this situation because it is defined through the usual inner product defined on matrix spaces $\langle A, B\rangle = \mathrm{tr}[A^T B] = \sum_{i,j} A_{i,j}B_{i,j}$.

2. Show that the equation $[A, B] = I$ has no solutions A, B, in the space of $n \times n$ matrices.

   Answer: We know that $\mathrm{tr}[AB] = \mathrm{tr}[BA]$, thus the trace of any commutator must be zero: $\mathrm{tr}[A, B] = \mathrm{tr}[AB] - \mathrm{tr}[BA] = 0$. Since $\mathrm{tr}[I] = n \neq 0$, we have that there are no A and B such that $[A, B] = I$.

3. How many multiplications does it take to evauate $\det A$ by using Guassian elimination to bring it into upper triangular form? How many multiplications does it take to evaluate $\det A$ by the formula

$$\det A = \sum_{p\in S_n} \sigma(p)a_{p_1,1} \cdots a_{p_n,n}?$$

   Answer: At each step of Guassian elimination, we must perform one multiplication for every entry that is modified. At the ith step, we must modify $n - i$ rows and $n - i + 1$ entries in each of these rows, so the number of multiplications to perform the Gaussian elimination is

$$\sum_{i=1}^{n-1}(n-i)(n-i+1) = \sum_{i=1}^{n-1} i(i+1) = \sum_{i=1}^{n-1} i^2 + i = \frac{1}{6}(n-1)n(2n-1) + \frac{1}{2}(n-1)n$$
$$= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{n}{6} + \frac{1}{2}n^2 - \frac{n}{2} = \frac{1}{3}n^3 - \frac{1}{3}n.$$

   Once the matrix is in upper triangular form, it takes $n - 1$ multiplications to compute the determinant, so the total number of multiplications is $\frac{n^3}{3} + \frac{2n}{3} - 1$.

   Computing $\det A$ through the Leibniz formula requires n multiplications (or $n-1$ depending on how you do it) for each term in the sum. Since we are summing over all permutations of n numbers, and we know that there are $n!$ distinct permutations of n elements, we have that this method requires $n(n!)$ multiplications, which is much larger than the above approach.

4. Show that the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 1 & \cdots & & 1 \\ 1 & 0 & 1 & & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & & 1 & 0 \end{pmatrix}$$

has a complete set of eigenvectors. What are it's eigenvalues? Compute $\det A$.

Answer: To find the eigenvalues of A, we must compute

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & -1 & \cdots & & -1 \\ -1 & \lambda & -1 & & -1 \\ \vdots & & \ddots & & \vdots \\ -1 & \cdots & & -1 & \lambda \end{pmatrix}.$$

Using the formula we derived on the last assignment for this kind of determinant gives that $\det(\lambda I - A) = (\lambda + 1)^{n-1}(\lambda - (n-1))$. This formula tells us a few things. First, we clearly have that if $\lambda = 0$, then $\det A = (-1)^n \det(-A) = (-1)^{n-1}(n-1)$.

Next, we see that $\det(\lambda I - A) = 0$ if $\lambda = -1$ or $n-1$, so the eigenvalues of -1 and $n-1$. The fact that $n-1$ is an eigenvalue tells us that it has at least one eigenvector, and that eigenvector will be linearly independent from the eigenvectors of eigenvalue -1. Since the multiplicity of the root is 1, we also know that there can only be one eigenvector of eigenvalue $n-1$.

For $\lambda = -1$, we see that

$$\lambda I - A = \begin{pmatrix} -1 & \cdots & -1 \\ \vdots & & \vdots \\ -1 & \cdots & -1 \end{pmatrix},$$

is a matrix with all rows being the same, so $\operatorname{rank}(-I - A) = 1$. The rank nullity theorem then tells us that $\operatorname{null}(-I - A) = n - 1$, so there exists a set of $n-1$ linearly independent vectors $v_2, \dots v_n$, such that $Av_j = -v_j$ for $j = 2, \dots n$. Since, these will be linearly independent from v_1, we have that $\{v_1, \dots, v_n\}$ will be a basis of eigenvectors.

Note: At this point we have completely answered the question. Sometimes all we need to is that there exists a basis of eigenvectors (this will tell us that our matrix is similar to a diagonal matrix, which tells us a lot about A). While it is possible to find an explicit basis of eigenvectors, it can be quite tedious, making the above approach more favorable in the (fairly common) case that we don't actually need the eigenvectors.

For illustrative purposes, I shall compute the eigenvectors here. For $\lambda = n-1$, we are looking for vectors such that

$$\begin{pmatrix} n-1 & -1 & \cdots & & -1 \\ -1 & n-1 & -1 & & -1 \\ \vdots & & \ddots & & \vdots \\ -1 & \cdots & & -1 & n-1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

Writing this another way, $\sum_{i \neq j} x_i = (n-1)x_j$ for $j = 1, \dots, n$. This is clearly satisfied by taking $x_j = c$ for all j, for some constant c. Since we know that there is only one linearly independent eigenvector for this eigenvalue, we have that that these are the only solution to this equation and we may take $v_1 = (1, \ldots, 1)^T$ is our eigenvector.

For $\lambda = -1$. We are looking for vectors such that

$$\begin{pmatrix} -1 & \cdots & -1 \\ \vdots & & \vdots \\ -1 & \cdots & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$
$$\Rightarrow x_1 + \cdots + x_n = 0.$$

The set of vectors in this hyperplane is $n-1$-dimensional. A convenient basis is

$$v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ and } v_n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix},$$

though there are many other choice of bases.

5. A Cauchy matrix is a matrix with entries

$$a_{ij} = \frac{1}{x_i - y_j},$$

where x and y are vectors with now shared elements and no repeated elements. Show that, if $x, y \in \mathbb{R}^n$, then

$$\det A = \frac{\prod_{j=2}^{n} \prod_{j=1}^{i-1} (x_i - x_j)(y_j - y_i)}{\prod_{i=1}^{n} \prod_{j=1}^{n} (x_i - y_i)}.$$

Answer: As with most of these kinds of problems, there are two main ways to do this, a polynomial-based approach, or a Guassian elimination-based approach.

Polynomial-based: Let

$$\begin{aligned} P(x_1, \ldots, x_n; y_1, \ldots, y_n) &= \prod_{i=1}^{n} \prod_{j=1}^{n} (x_i - y_j) \det A \\ &= \det \begin{pmatrix} \frac{\prod_{j=1}^{n}(x_1 - y_j)}{x_1 - y_1} & \cdots & \frac{\prod_{j=1}^{n}(x_1 - y_j)}{x_1 - y_n} \\ \vdots & & \vdots \\ \frac{\prod_{j=1}^{n}(x_n - y_j)}{x_n - y_1} & \cdots & \frac{\prod_{j=1}^{n}(x_n - y_j)}{x_n - y_n} \end{pmatrix}, \text{ by multilinearity} \\ &= \det \begin{pmatrix} \prod_{j \neq 1}(x_1 - y_j) & \cdots & \prod_{j \neq n}(x_1 - y_j) \\ \vdots & & \vdots \\ \prod_{j \neq 1}(x_n - y_j) & \cdots & \prod_{j \neq n}(x_n - y_j) \end{pmatrix} \end{aligned}$$

Written in this form, we can see that P is a polynomial of $\{x_i\}$ and $\{y_i\}$. The Leibniz formula tells us that it will be a polynomial of degree at most $n(n-1)$. We see that P must zero if $x_i = x_j$ or $y_i = y_j$, because then A has two equal rows/columns, which will imply that $\det A = 0$. Thus $P = Q(x_1, \ldots, x_n; y_1, \ldots, y_n) \prod_{j=2}^n \prod_{j=1}^{i-1} (x_i - x_j)(y_i - y_j)$, for some polynomial Q. Counting degrees tells us that Q is a constant.

To find the value of the constant Q we may plug in a specific choice of x and y. One nice choice is $x_i = \frac{1}{2} + it$ and $y_j = -\frac{1}{2} + jt$. Then $A_{ij} = \frac{1}{1+(i-j)t}$. As $t \to \infty$, A will approach the identity, and so $\det A \to 1$. Plugging these values into our formula for P tells us that

$$
\begin{aligned}
Q &= \lim_{t\to\infty} \frac{\prod_{i=1}^n \prod_{j=1}^n (x_i - y_j) \det A}{\prod_{j=2}^n \prod_{j=1}^{i-1} (x_i - x_j)(y_i - y_j)} \\
&= \lim_{t\to\infty} \frac{\prod_{i=1}^n \prod_{j=1}^n (1 + (i-j)t)}{\prod_{j=2}^n \prod_{j=1}^{i-1} (i-j)^2 t^2} \\
&= \lim_{t\to\infty} \frac{\prod_{i\neq j} (i-j)}{\prod_{j=2}^n \prod_{j=1}^{i-1} (i-j)^2} \frac{t^{n^2-n}}{t^{n^2-n}} \\
&= \frac{\prod_{j=2}^n \prod_{j=1}^{i-1} \left(-(i-j)^2\right)}{\prod_{j=2}^n \prod_{j=1}^{i-1} (i-j)^2} \\
&= (-1)^{n^2-n}.
\end{aligned}
$$

If we now isolate $\det A$ from P, we have that

$$
\det A = \frac{(-1)^{n^2-n} \prod_{j=2}^n \prod_{j=1}^{i-1} (x_i - x_j)(y_i - y_j)}{\prod_{i=1}^n \prod_{j=1}^n (x_i - y_j)} = \frac{\prod_{j=2}^n \prod_{j=1}^{i-1} (x_i - x_j)(y_j - y_i)}{\prod_{i=1}^n \prod_{j=1}^n (x_i - y_j)}.
$$

Guassian elimination-based:

$$
\begin{aligned}
\det A &= \det \begin{pmatrix} \frac{1}{x_1-y_1} & \cdots & \frac{1}{x_1-y_n} \\ \vdots & & \vdots \\ \frac{1}{x_n-y_1} & \cdots & \frac{1}{x_n-y_n} \end{pmatrix} \\
&= \prod_{i=1}^{n} \frac{1}{x_i-y_1} \det \begin{pmatrix} 1 & \frac{x_1-y_1}{x_1-y_2} & \cdots & \frac{x_1-y_1}{x_1-y_n} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{x_n-y_1}{x_n-y_2} & \cdots & \frac{x_n-y_1}{x_n-y_n} \end{pmatrix} \\
&= \prod_{i=1}^{n} \frac{1}{x_i-y_1} \det \begin{array}{c} \begin{array}{cccc} & c_2-c_1 & & c_n-c_1 \end{array} \\ \begin{pmatrix} 1 & \frac{y_2-y_1}{x_1-y_2} & \cdots & \frac{y_n-y_1}{x_1-y_n} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{y_2-y_1}{x_n-y_2} & \cdots & \frac{y_n-y_1}{x_n-y_n} \end{pmatrix} \end{array} \\
&= \prod_{i=1}^{n} \frac{1}{x_i-y_1} \prod_{j=2}^{n} (y_j-y_1) \det \begin{pmatrix} 1 & \frac{1}{x_1-y_2} & \cdots & \frac{1}{x_1-y_n} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{1}{x_n-y_2} & \cdots & \frac{1}{x_n-y_n} \end{pmatrix} \\
&= \prod_{i=1}^{n} \frac{1}{x_i-y_1} \prod_{j=2}^{n} (y_j-y_1) \det \begin{array}{c} \\ r_2-r_1 \\ \\ r_n-r_1 \end{array} \begin{pmatrix} 1 & \frac{1}{x_1-y_2} & \cdots & \frac{1}{x_1-y_n} \\ 0 & \frac{x_1-x_n}{(x_2-y_2)(x_1-y_2)} & \cdots & \frac{x_1-x_2}{(x_2-y_n)(x_1-y_n)} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{x_1-x_n}{(x_n-y_2)(x_1-y_2)} & \cdots & \frac{x_1-x_n}{(x_n-y_n)(x_1-y_n)} \end{pmatrix} \\
&= \prod_{i=1}^{n} \frac{1}{x_i-y_1} \prod_{j=2}^{n} (y_j-y_1) \prod_{j=2}^{n} (x_1-x_j) \prod_{i=1}^{n} \frac{1}{x_1-y_j} \det \begin{array}{c} \\ r_2/(x_2-1) \\ \\ r_2/(x_n-1) \end{array} \begin{array}{c} \begin{array}{cccc} & c_2(x_1-y_2) & & c_n(x_1-y_n) \end{array} \\ \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \frac{1}{x_2-y_2} & \cdots & \frac{1}{x_2-y_n} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{1}{x_n-y_2} & \cdots & \frac{1}{x_n-y_n} \end{pmatrix} \end{array} \\
&= \frac{\prod_{j=2}^{n} (y_1-y_j)(x_j-x_1)}{\prod_{i=1}^{n} (x_i-y_1)(x_1-y_j)} \det \begin{pmatrix} \frac{1}{x_2-y_2} & \cdots & \frac{1}{x_2-y_n} \\ \vdots & & \vdots \\ \frac{1}{x_n-y_2} & \cdots & \frac{1}{x_n-y_n} \end{pmatrix} \\
&\vdots \quad \text{inductively applying the same argument gives} \\
&= \frac{\prod_{j=2}^{n} \prod_{j=1}^{i-1} (x_i-x_j)(y_j-y_i)}{\prod_{i=1}^{n} \prod_{j=1}^{n} (x_i-y_j)}.
\end{aligned}
$$

6. Consider the $n \times n$ matrix

$$
A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & & & \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.
$$

Show that the eigenvalues of A are the form

$$
\lambda_{k+1} = a_1 \omega_k^0 + \cdots + a_n \omega_k^{n-1}, 0 \le k \le n-1,
$$

where $\omega_k = e^{2\pi i k/n}$.

Answer: We note that $\omega_k^n = e^{2\pi i k} = 1$, so

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & & & \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix} \begin{pmatrix} \omega_k^0 \\ \vdots \\ \omega_k^{n-1} \end{pmatrix} = \begin{pmatrix} a_1\omega_k^0 + \cdots + a_n\omega_k^{n-1} \\ a_1\omega_k^1 + \cdots + a_n\omega_k^n \\ \vdots \\ a_1\omega_k^{n-1} + \cdots + a_n\omega_k^{2n-2} \end{pmatrix} = (a_1\omega_k^0 + \cdots + a_n\omega_k^{n-1}) \begin{pmatrix} \omega_k^0 \\ \vdots \\ \omega_k^{n-1} \end{pmatrix}.$$

Thus $v_k = (1, \ldots, \omega_k^{n-1})$ is an eigenvector with eigenvalue $\lambda_{k+1} = a_1\omega_k^0 + \cdots + a_n\omega_k^{n-1}$ for every k. These are all of the eigenvalues because we have found n eigenvectors. If $a_1, \ldots, a_n$ are all different, then the eigenvalues will all be distinct, and so the the eigenvectors will be linearly independent. Since the eigenvectors are independent of $a_1, \ldots, a_n$, we have found n linearly independent eigenvectors, and so have all of the eigenvalues, even in the case that A is degenerate.

We may also show that the eigenvectors are linearly independent explicitly by taking the dot product of any pair of eigenvectors

$$\langle v_k, v_l \rangle = \sum_{j=1}^{n} e^{2\pi i \frac{k-l}{n}}.$$

If $k \neq l$, then we are summing n distinct nth roots of unity, which gives zero. (There is a very nice geometric proof of this fact that you can look up if you have not seen it already.) Thus if we assemble the eigenvectors into a matrix

$$U = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \omega_1^{n-1} & \cdots & \omega_n^{n-1} \end{pmatrix},$$

we have that $U^*U = nI$. Thus $\det U \neq 0$, and so the eigenvectors are linearly independent.