1. Let $(\cdot, \cdot)$ be a complex inner product on a linear space X and $\|x\| = \sqrt{(x,x)}$. Show the following results.

   (a) $|(x,y)| \le \|x\|\|y\|, \forall x, y \in X$.

   Answer: Let $t \in \mathbb{C}$, by the definition of $\|\cdot\|$, we know that

$$
\begin{aligned}
0 \le \|x+ty\|^2 = (x+ty, x+ty) &= (x,x) + \bar{t}(x,y) + t\overline{(x,y)} + |t|^2(y,y) \\
&= \|x\|^2 + 2\Re\bar{t}(x,y) + |t|^2\|y\|^2.
\end{aligned}
$$

   If we suppose that $y \neq 0$, and let $t = -(x,y)/\|y\|^2$, then we have

$$
\begin{aligned}
2|(x,y)|/\|y\|^2 &\le \|x\|^2 + |(x,y)|^2\|y\|^2/\|y\|^4 \\
&\Rightarrow |(x,y)|/\|y\|^2 \le \|x\|^2 \\
&\Rightarrow \|(x,y)\|^2 \le \|x\|^2\|y\|^2.
\end{aligned}
$$

   Noting that $y = 0$ implies that both sides are 0, we have proved the Cauchy-Schwarz inequality for all $x, y \in X$.

   (b) $\|x+y\| \le \|x\| + \|y\|, \forall x, y \in X$.

   Answer: For any $x, y \in X$ we have that

$$\|x+y\|^2 = (x+y, x+y) = \|x\|^2 + 2\Re(x,y) + \|y\|^2$$

   applying the Cauchy-Schwarz inequality

$$\|x+y\|^2 \le \|x\|^2 + 2|(x,y)| + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \le (\|x\| + \|y\|)^2.$$

   Taking square roots gives the desired result.

2. Let X be the space of polynomials on the interval $[-1, 1]$ with real coefficients and of degree $\le n$. We define the inner product

$$(p,q) = \int_{-1}^{1} p(x)q(x)dx, \quad p, q \in X.$$

   For $l = 0, 1, 2, \ldots$ define the polynomial of degree l

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

   te $P_l's$ are called Legendre polynomials. Set

$$q_l(x) = \sqrt{\frac{2l+1}{2}}P_l(x).$$

   (a) Show that $q_l,\ l = 0, \ldots, n$ is an orthonormal basis for X.

   Answer: Letting $g_l(x) = (x^2-1)^l$, we have that

$$
\begin{aligned}
Dg_l(x) &= l(x^2-1)^{l-1}2x \\
\Rightarrow (x^2-1)Dg_l(x) &= 2lxg_l
\end{aligned}
$$

Taking $l+1$ derivatives of both sides gives

$$\begin{aligned}(x^2-1)D^{l+2}g_l + 2x(l+1)D^{l+1}g_l + l(l+1)D^l g_l &= 2lxD^{l+1}g_l + 2l(l+1)D^l g_l \\ \Rightarrow (x^2-1)D^2 q_l + 2xDq_l &= l(l+1)q_l \\ \Rightarrow ((x^2-1)q_l')' &= l(l+1)q_l. \qquad (1)\end{aligned}$$

Multiplying (1) by q_k and subtracting the equivalent expression with l and k swapped gives

$$\begin{aligned} l(l+1)q_lq_k - k(k+1)q_lq_k &= ((x^2-1)q_l')'q_k - ((x^2-1)q_k')'q_l \\ [l(l+1)-k(k+1)]\, q_lq_k &= ((x^2-1)q_l'q_k)' - (x^2-1)q_l'q_k' - ((x^2-1)q_k'q_l)' + (x^2-1)q_l'q_k' \\ &= \big((x^2-1)(q_l'q_k - q_lq_k')\big)' \end{aligned}$$

Integrating from $x=-1$ to 1, we have that

$$[l(l+1)-k(k+1)]\,(q_l,q_k) = \int_{-1}^1 \big((x^2-1)(q_l'q_k - q_lq_k')\big)'\,dx = (x^2-1)(q_l'q_k - q_lq_k')\big|_{x=-1}^1 = 0,$$

thus if $l \neq k$, we have that $(q_l,q_k)=0$.
To finish our proof, we compute

$$\begin{aligned}(q_l,q_l) &= c_l^2\int_{-l}^1 (D^l(x^2-1)^l)(D^l(x^2-1)^l)dx \\ &= c_l^2\left[(D^{l+1}(x^2-1)^l)(D^l(x^2-1)^l)\Big|_{-1}^1 - \int_{-l}^1 (D^{l+1}(x^2-1)^l)(D^{l-1}(x^2-1)^l)dx\right] \\ &= c_l^2\left[0 - 0 + \cdots + (-1)^l\int_{-1}^1 (D^{2l}(x^2-1)^l)(x^2-1)^l dx\right]\end{aligned}$$

Noting that $(x^2-1)^l$ is a polynomial of degree $2l$, we have that $D^{2l}(x^2-1)^l = (2l)!$. This gives that

$$\begin{aligned}(q_l,q_l) &= \frac{(2l)!}{2^{2l}(l!)^2}\frac{2l+1}{2}\int_{-1}^1 (1-x^2)^l dx \\ &= \frac{(2l)!}{2^{2l}(l!)^2}\frac{2l+1}{2}\int_{-\pi/2}^{\pi/2} (1-\sin^2(\theta))^l\cos(\theta)d\theta \\ &= \frac{(2l)!}{2^{2l}(l!)^2}\frac{2l+1}{2}\int_{-\pi/2}^{\pi/2} \cos^{2l+1}(\theta)d\theta \\ &= \frac{(2l)!}{2^{2l}(l!)^2}\frac{2l+1}{2}\left[\sin\theta\cos^{2l}\theta\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}(-2l)\sin^2\theta\cos^{2l-1}\theta d\theta\right] \\ &= \frac{(2l)!}{2^{2l}(l!)^2}\frac{2l+1}{2}2l\left[0 + \int_{-\pi/2}^{\pi/2}\cos^{2l-1}\theta d\theta - \int_{-\pi/2}^{\pi/2}\cos^{2l+1}\theta d\theta\right].\end{aligned}$$

Letting $I_l = \int_{-\pi/2}^{\pi/2}\cos^{2l+1}(\theta)d\theta$, we have shown that

$$\begin{aligned} I_l &= 2lI_{l-1} - 2lI_l \\ \Rightarrow I_l &= \frac{2l}{2l+1}I_{l-1} = \cdots = \frac{2l}{2l+1}\frac{2l-2}{2l-1}\cdots\frac{2}{3}I_0 \\ \Rightarrow I_l &= \frac{(2l)!!}{(2l+1)!!}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = \frac{2^{2l}(l!)^2}{(2l+1)!}2.\end{aligned}$$

Plugging this in, we find that

$$(q_l, q_l) = \frac{(2l)!}{2^{2l}(l!)^2}\frac{2l+1}{2}\frac{2^{2l}(l!)^2}{(2l+1)!}2 = \frac{2l+1}{2}\frac{2}{2l+1} = 1.$$

Thus $\{q_l\}$ forms an orthonormal system. Since there are n of them, they form a basis for X.

(b) Write the Legendre polynomials $\{P_l\}$ in the standard basis.

Answer: Direct computation gives that

$$\begin{aligned} P_l(x) &= D^l(x^2-1)^l = D^l \sum_{k=0}^{l} \binom{l}{k} x^{2l-2k}(-1)^k \\ &= \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \binom{l}{k} (2l-2k)\cdots(l-2k+1)x^{l-2k} \\ &= \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \binom{l}{k} \binom{2l-2k}{l} x^{l-2k}. \end{aligned}$$

(c) Show that $\{q_l\}$ is the orthonormal basis for polynomials of degree with complex coefficients and inner product on $[-1,1]$

$$(p,q) = \int_{-1}^{1} p(x)\overline{q(x)}dx.$$

Answer: Since the q_l's have real coefficients, the result $(q_l, q_k) = \delta_{l,k}$ still holds. As we have n vectors, we still have that $\{q_l\}$ form an orthonormal basis.

3. Let $T_n(x) = \cos(n \arccos x)$, $n = 0, 1, \ldots$ be the Chebyshev polynomials defined on $[-1,1]$.

(a) Show that the T_n's are polynomials.

Answer: Note that

$$\begin{aligned} T_{n+1}(x) &= \cos((n+1)\arccos x) = \cos(n\arccos x + \arccos x) \\ &= \cos(\arccos x)\cos(n\arccos x) - \sin(\arccos x)\sin(n\arccos x) \\ &= xT_n(x) - \sin(\arccos x)(\sin((n-1)\arccos x)\cos(\arccos x) \\ &\qquad + \cos((n-1)\arccos x)\sin(\arccos x) \\ &= xT_n(x) - \sin^2(\arccos x)\cos((n-1)\arccos x) \\ &\qquad - \sin(\arccos x)\cos(\arccos x)\sin((n-1)\arccos x) \\ &= xT_n(x) - T_{n-1}(x) + \cos(\arccos x)(\cos(\arccos x)\cos((n-1)\arccos x) \\ &\qquad - \sin(\arccos x)\sin((n-1)\arccos x)) \\ &= xT_n(x) - T_{n-1}(x) + x\cos((n-1)\arccos x) \\ &= 2xT_n(x) - T_{n-1}(x). \end{aligned}$$

Noting that $T_0(x) = 1$ and $T_1(x) = x$, this recursion relation gives that the Chebyshev polynomials are indeed polynomials.

(b) Let X be the space of real polynomials of degree $\leq N$ on $(-1,1)$ with inner products

$$(p,q) = \int_{-1}^{1} p(x)q(x)\frac{dx}{\sqrt{1-x^2}}.$$

Let $t_n = \begin{cases} \sqrt{\frac{2}{\pi}}T_n & n>0 \\ \frac{1}{\sqrt{\pi}}T_0 & n=0 \end{cases}$. Show that $t_0,\ldots,t_n$ is an orthonormal basis for X.

Answer: First note that

$$\begin{aligned}(T_n,T_m) &= \int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2}\cos(n\theta)\cos(m\theta)\frac{-\sin\theta d\theta}{\sin\theta} \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos((m-n)\theta)+\cos((m+n)\theta)d\theta.\end{aligned}$$

If $n \neq m$, we have that the inner product is

$$(T_n,T_m) = \frac{1}{2}\left[\frac{\sin((m-n)\theta)}{m-n} + \frac{\sin((m+n)\theta)}{m+n}\right]_{-\pi/2}^{\pi/2} = 0.$$

If $n = m > 0$, we have that

$$\begin{aligned}(T_n,T_n) &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta + 0 = \frac{\pi}{2} \\ &\Rightarrow (t_n,t_n) = 1.\end{aligned}$$

Finally, if $n = m = 0$, we have that

$$(t_0,t_0) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi}{\pi} = 1.$$

Combining these results and noting that $\dim X = N+1 = |\{t_n\}_{n=0}^{N}|$, we have that $\{t_n\}_{n=0}^{N}$ is an orthonormal basis for X.

4. Let $(X,(\cdot,\cdot))$ be a real inner product space with an induced norm $\|x\| = \sqrt{(x,x)}$.

(a) Show that if $\|\cdot\|$ arises from an inner product, then the parallelogram law holds:

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2.$$

Answer: We proceed by direct computation:

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= (x+y,x+y) + (x-y,x-y) \\ &= (x,x)+2(x,y)+(y,y)+(x,x)-2(x,y)+(y,y) = 2\|x\|^2+2\|y\|^2.\end{aligned}$$

(b) Conversely, show that if the parallelogram law holds for $(X,\|\cdot\|)$, then

$$(x,y) := \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

is an inner product on X and $\|x\| = \sqrt{(x,x)}$.

Answer: First note that $(x,y) = \frac{\|x+y\|^2 - \|x-y\|^2}{4} = (y,x)$, so $(\cdot,\cdot)$ is symmetric.
To show that $(\cdot,\cdot)$ is linear, note that, since the parallelogram law holds for $\|\cdot\|$, we have

$$\begin{aligned}
&\|x+y+z\|^2 \\
=&\frac{1}{2}[\|x+y+z\|^2 + \|x-y+z\|^2] + \frac{1}{2}[\|x+y+z\|^2 + \|y-x+z\|^2] - \frac{1}{2}\|x-y+z\|^2 - \frac{1}{2}\|y-x+z\|^2 \\
=&\|x+z\|^2 + \|y\|^2 + \|y+z\|^2 + \|x\|^2 - \frac{1}{2}\|x-y+z\|^2 - \frac{1}{2}\|y-x+z\|^2
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|x+y-z\|^2 \\
=&\|x-z\|^2 + \|y\|^2 + \|y-z\|^2 + \|x\|^2 - \frac{1}{2}\|x-y-z\|^2 - \frac{1}{2}\|y-x-z\|^2 \\
=&\|x-z\|^2 + \|y\|^2 + \|y-z\|^2 + \|x\|^2 - \frac{1}{2}\|y+z-x\|^2 - \frac{1}{2}\|x+z-y\|^2
\end{aligned}$$

Subtracting these expressions gives that

$$\begin{aligned}
\|x+y+z\|^2 - \|x+y-z\|^2 &= \|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2 \\
\Rightarrow \frac{\|x+y+z\|^2 - \|x+y-z\|^2}{4} &= \frac{\|x+z\|^2 - \|x-z\|^2}{4} + \frac{\|y+z\|^2 - \|y-z\|^2}{4} \\
\Rightarrow (x+y,z) &= (x,z) + (y,z),
\end{aligned}$$

so $(\cdot,\cdot)$ is linear.
To show that $(\cdot,\cdot)$ is homogeneous, we start by noting that for any $n \in \mathbb{Z}$, we have

$$(nx,y) = ((n-1)x,y) + (x,y) = ((n-2)x,y) + 2(x,y) = \cdots = n(x,y).$$

Also

$$\begin{aligned}
\left(\frac{n}{n}x,y\right) &= n\left(\frac{1}{n}x,y\right) \\
\Rightarrow \left(\frac{1}{n}x,y\right) &= \frac{1}{n}(x,y) \\
\Rightarrow \left(\frac{m}{n}x,y\right) &= \frac{m}{n}(x,y),
\end{aligned}$$

so $(\lambda x,y) = \lambda(x,y)$ for all $\lambda \in \mathbb{Q}^+$.
Also note that

$$\begin{aligned}
(x-x,y) &= (x,y) + (-x,y) \\
\Rightarrow \frac{\|0+y\|^2 - \|0-y\|^2}{4} &= (x,y) + (-x,y) \\
\Rightarrow 0 &= (x,y) + (-x,y) \\
\Rightarrow (-x,y) &= -(x,y),
\end{aligned}$$

so $(\lambda x,y) = \lambda(x,y)$ for all $\lambda \in \mathbb{Q}$. Noting that $(\cdot,\cdot)$ is a difference between continuous functions, so we may take limits to show homogeneity over all real numbers.

Finally note that

$$(x,x) = \frac{\|2x\|^2 - \|0\|^2}{4} = \|x\|^2.$$

so $(\cdot,\cdot)$ induces the desired norm. This also shows that $(x,x) > 0$ for all $x \neq 0$ and if $(x,x) = 0$, then $\|x\| = 0$, so $x = 0$.

(c) For $X = \mathbb{R}^n$, set

$$\|x\|_m := \max_i |x_i|, \quad x = (x_1, \ldots, x_n).$$

Show that $\|\cdot\|_m$ is a norm on X, which does not arise from an inner product on X.

Answer: Note $\|x\|_m \geq 0$ for all $x \in X$. If $\|x\|_m = 0$, then we have that $x_i = 0$ for all i, so $x = 0$.
Also

$$\|\alpha x\|_m = \max_i |\alpha x_i| = \max_i |\alpha||x_i| = |\alpha| \max_i |x_i| = |\alpha| \|x\|_m.$$

Finally note that

$$\|x+y\|_m = \max_i |x_i + y_i| \leq \max_i |x_i| + |y_i| \leq \max_i |x_i| + \max_j |y_j| = \|x\|_m + \|y\|_m,$$

because the absolute value satisfies the triangle inequality. Thus $\|\cdot\|_m$ satisfies all of the requirements of a norm.
To show that $\|\cdot\|$ does not arise from an inner product, we show that it does not obey the parallelogram law.
Let $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$. Then

$$\|x\|_m = \|y\|_m = \|x+y\|_m = \|x-y\|_m = 1$$
$$\Rightarrow \|x+y\|_m^2 + \|x-y\|_m^2 = 1 + 1 = 2 \neq 4 = 2\|x\|_m^2 + 2\|y\|_m^2,$$

so $\|\cdot\|_m$ does not obey the parallelogram law, which by a), means it does not come from an inner product.

(d) Let X be the linear space of continuous functions on $[0,1]$ and set

$$\|f\|_{L^1} = \int_0^1 |f(t)| dt.$$

Show that $\|f\|_{L^1}$ is a norm on X, which does not come from an inner product.

Answer: For any $\|f\|_{L^1} = \int_0^1 |f(t)| dt \geq 0$. Also, if $\|f\|_{L^1} = 0$, then we know that $f = 0$ almost everywhere. Since we only working with continuous functions, we have that $f \equiv 0$.
We also compute

$$\|\alpha f\|_{L^1} = \int_0^1 |\alpha f(t)| dt = |\alpha| \int_0^1 |f(t)| dt = |\alpha| \|f\|_{L^1}.$$

Finally we note that

$$\|f+g\|_{L^1} = \int_0^1 |f(t)+g(t)|dt \leq \int_0^1 |f(t)| + |g(t)|dt \leq \|f\|_{L^1} + \|g\|_{L^1},$$

since the absolute value satisfies the inner product.
Thus $\|\cdot\|_{L^1}$ satisfies the definition of a norm on X. It is good to note that in both c) and d), the hardest part of checking that we had a norm was proving the triangle inequality. Homogeneity and non-negativity could be checked directly, but proving the triangle inequality required knowing that some other function satisfied the triangle inequality. While the proof of the triangle inequality is sometimes done differently, it is generally true that it is the hardest part of checking that something is a norm.
To show that $\|\cdot\|_{L^1}$ is not induced by any inner product, let $f(t) = t$ and $g(t) = 2$. Then

$$\|f\|_{L^1} = \int_0^1 tdt = \frac{1}{2}, \|g\|_{L^1} = \int_0^1 2dt = 2.$$

Also, since $|f(t)+g(t)| = f(t)+g(t)$ and $|g(t)-f(t)| = g(t)-f(t)$, we have that

$$\|f+g\|_{L^1} = \|f\|_{L^1} + \|g\|_{L^1} = \frac{5}{2}, \|g-f\|_{L^1} = \|g\|_{L^1} - \|f\|_{L^1} = \frac{3}{2}.$$

Thus

$$\|f+g\|^2_{L^1} + \|g-f\|^2_{L^1} = \frac{34}{4} \neq \frac{9}{2} = 2\|f\|^2_{L^1} + \|g\|^2_{L^1},$$

so $\|\cdot\|_{L^1}$ does not obey the parallelogram law, so cannot come from an inner product.

5. Let $(X, \|\cdot\|)$ be a finite dimensional normed vector space as in problem 4, $\dim X = n < \infty$. For $l \in X^1$, the dual space of X, define

$$\|l\| = \sup_{\|x\|\leq 1} |l(x)|.$$

(a) Show that $\|l\|$ is a norm on X'.

Answer: It is clear that $\|l\| \geq 0$ for all $l \in X'$. If $\|l\| = 0$, then $l(x) = 0$ for all $x \in X$, so $l = 0_{X'}$.
We also see that

$$\|\alpha l\| = \sup_{\|x\|\leq 1} |(\alpha l)(x)| = |\alpha| \sup_{\|x\|\leq 1} |l(x)| = |\alpha|\|l\|.$$

Finally, if $l_1, l_2 \in X'$, then

$$\|l_1 + l_2\| = \sup_{\|x\|\leq 1} |l_1(x) + l_2(x)| \leq \sup_{\|x\|\leq 1} |l_1(x)| + \sup_{\|x\|\leq 1} |l_2(x)| = \|l_1\| + \|l_2\|.$$

So $\|l\|$ is indeed a norm on X'.

(b) Show that there is a basis $x_1, \ldots, x_n$ for X and a basis $l_1, \ldots, l_n$. For X' such that

- $\|x_i\| = 1$, $i = 1, \ldots, n$.
- $\|l_i\| = 1$, $i = 1, \ldots, n$.

- $l_i(x_j) = \delta_{i,j}$, $1 \le i, j \le n$.

Answer: Let $\{y_i\}$ be a basis for X. Define $x_i = y_i/\|y_i\|$ for $i = 1, \dots, n$, then $\{x_i\}$ is a normalized basis for X.
Define a multi-linear functional H, which satisfies

$$H(z_1, \dots, z_n) = 0, \text{ if } z_i = z_j \text{ for any } i \neq j$$

and

$$H(x_1, \dots, x_n) = 1.$$

We note that if we map all vectors to their coordinate vectors under the basis $\{x_i\}$, then H is mapped to the determinant on $\mathbb{R}^n$.
For any j, we will define

$$l_j(x) = H(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n).$$

We see that $l_j(x_i) = \delta_{i,j}$.
Using the fact that the determinant can be interpreted as the signed volume of a hyper-parallelepiped formed by the involved vectors, we have that

$$|l_j(x)| = |H(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n)| \le \|x_1\| \cdots \|x_{j-1}\|\|x\|\|x_{j+1}\| \cdots \|x_n\| = \|x\|,$$

so $\|l_j\| \le 1$. Also noting that $\|l_j(x_j)\| = 1 = \|x_j\|$, we have equality $\|l_j\| = 1$.

6. Let $(X, \|\cdot\|)$ be a normed linear space and let A be a linear mapping from X to X. In analogy to the above norm, set

$$\|A\| = \sup_{\|x\| \le 1} \|Ax\|.$$

(a) Show that $\|A\|$ is a norm on $\mathcal{L}(X)$.

Answer: Clearly $\|A\| \le 0$ for all A. If $\|A\| = 0$, then $\|Ax\| = 0$ for all $x \in X$, and so $Ax = 0$ for all $x \in X$, and so $A = 0_{\mathcal{L}(X)}$.
We also see that

$$\|\alpha A\| = \sup_{\|x\| \le 1} \|\alpha A x\| = |\alpha| \sup_{\|x\| \le 1} \|Ax\| = |\alpha| \|A\|.$$

We also have that for any $A, B \in mcL(X)$,

$$\|A + B\| = \sup_{\|x\| \le 1} \|(A+B)x\| \le \sup_{\|x\| \le 1} (\|Ax\| + \|Bx\|) \le \sup_{\|x\| \le 1} \|Ax\| + \sup_{\|x\| \le 1} \|Bx\| = \|A\| + \|B\|,$$

so $\|A\|$ is a norm on $\mathcal{L}(X)$.

(b) Let $(X, \|\cdot\|)$ be a finite dimensional inner product space, $\dim X = n < \infty$ and Y be a subspace of X. Show that there is a projection P_Y of X onto Y such that $\|P_Y\| \le n$.

Answer: Let $\{x_i\}$ be a normalized basis for X, where $x_1, \dots, x_k \in Y$ and the rest are not.

Let $\{l_i\}$ be the corresponding dual basis described in 5b). Then we shall define

$$P_Y(x) = \sum_{i=1}^{k} l_i(x)x_i.$$

We see that

$$P_Y(P_Y(x)) = \sum_{j=1}^{k} l_j\left(\sum_{i=1}^{k} l_i(x)x_i\right)x_j = \sum_{1\leq i,j\leq k} l_i(x)l_j(x_i)x_j = \sum_{j=1}^{k} l_j(x)x_j = P_Y(x),$$

so P_Y is indeed a projection. The definition of $\{x_i\}$ gives that it is a projection onto Y. Finally, for any $x \in X$ we have

$$\|P_y(x)\| \leq \sum_{i=1}^{k} \|l_i(x)x_i\| = \sum_{i=1}^{k} |l_i(x)|\|x_i\| \leq \sum_{i=1}^{k} \|l_i\|\|x\| \leq \sum_{i=1}^{k} \|x\| = k\|x\| \leq n\|x\|,$$

so $\|P_Y\| \leq n$, as desired.

7. Find the adjoint of the differential operator

$$L(f)(x) = \frac{df}{dx}(x),$$

in the space of smooth periodic function on $(0, 2\pi)$ with the inner product

$$(f, g) = \int_0^{2\pi} f(x)g(x)(2 + \sin x)dx.$$

Answer: We proceed by direct calculation. let f, g be smooth periodic functions on $(0, 2\pi)$. Then

$$\begin{aligned}
(L(f), g) &= \int_0^{2\pi} \frac{df}{dx}(x)g(x)(2 + \sin x)dx \\
&= f(x)g(x)(2 + \sin x)|_0^{2\pi} - \int_0^{2\pi} f(x)\frac{d}{dx}[g(x)(2 + \sin x)]dx \\
&= -\int_0^{2\pi} f(x)\left[\frac{dg}{dx}(2 + \sin x) + g(x)\cos x\right]dx, \text{ since } f, g, \text{ and } \sin \text{ are periodic.} \\
&= -\left(f, \frac{dg}{dx}(2 + \sin x) + g(x)\cos x\right)
\end{aligned}$$

Thus the adjoint of L is

$$L'(g)(x) = \frac{dg}{dx}(2 + \sin x) + g(x)\cos x.$$