Linear Algebra

Problem set #6

Due: Nov 1, 2018

Oct 26, 2018

1. Let A be an $n \times n$ real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ which are all real and positive. How many real matrices B satisfy the matrix equation $A = B^2$?

2. A matrix A is said to be monotone if for $x = (x_1, \ldots, x_n)^T$, $Ax \geq 0$ (ie all the entries of Ax are ≥ 0) implies $x \geq 0$ (ie again all the entries of x are ≥ 0.

   (a) Show that A is invertible

   (b) Show that A is monotone if and only if A^{-1} exists and all its entries are non-negative.

3. Let A and B be rectangular matrices of sizes $n \times m$ and $m \times n$ respectively. Prove that the non-zero eigenvalues of AB and BA are the same. Express the corresponding eigenvectors of AB in terms of those of BA. Show that if $\lambda + AB$ is invertible, $\lambda \neq 0$ (and so $\lambda + BA$ is invertible), then

$$\frac{\lambda}{\lambda + AB} + A \frac{1}{\lambda + BA} B = I$$

4. Let C be an $n \times n$ matrix.

(a) Show that

$\operatorname{tr} C = 0$ if and only if $C = SDS^{-1}$ for some invertible S and D is zero on the diagonal i.e. $D_{ii} = 0$, $i = 1, \dots, n$.

(b) Show that

$\operatorname{tr} C = 0$ if and only if $C = AB - BA$ for some $n \times n$ matrices A, B.

5. Show that if A is a square matrix with complex entries and $A^n = I$ for some $n > 0$, then A has n linearly independent eigenvectors.

6. A square matrix S is stochastic if all its elements are non-negative and the sum of the elements in each column is 1, i.e. $\sum_{i=1}^{n} S_{ij} = 1$, $1 \le j \le n$.

Show that

(a) $\lambda = 1$ is an eigenvalue of S

(b) All eigenvalues λ_i of S lie in the closed disc in $\mathbb{C}$, i.e. $|\lambda_i| \le 1$.