Linear Algebra
Problem set #1

Due: Sep 27, 2018

Sep 20, 2018

1. Let X be the space of polynomials of degree $< n$, and let Y be the set of polynomials that are zero at $t_1, \ldots, t_j$, $j < n$, $t_i \in \mathbb{R}$. Determine $\dim Y$ and $\dim X/Y$

2. In Theorem 6 (Lax, p 17) take the interval I to be $[-1, 1]$, and take n to be 3. Choose the 3 points to be $t_1 = -a$, $t_2 = 0$, and $t_3 = a$

(i) Determine the weights m_1, m_2, m_3 so that

$$\int_I p(t)\,dt = m_1 p(t_1) + m_2 p(t_2) + m_3 p(t_3) \qquad (*)$$

holds for all polynomials of degree < 3.

(ii) Show that for $a > \sqrt{1/3}$, all three weights are positive

(iii) Show that for $a = \sqrt{3/5}$, $(*)$ holds for all polynomials of degree < 6

3. Let P_2 be the linear space of all polynomials

$$p(x) = a_0 + a_1 x + a_2 x^2$$

with real coefficients and degree ≤ 2. Let ξ_1, ξ_2, ξ_3 be three distinct real numbers, and then define

$$l_j = p(\xi_j) \quad , \quad j = 1, 2, 3 .$$

(a) Show that l_1, l_2, l_3 is a basis for P_2'

(b) (1) Suppose $\{e_1, \ldots, e_n\}$ is a basis for a vector space V. Show there exist linear functions $\{l_1, \ldots, l_n\}$ in V' defined by

$$l_i(e_j) = 1 \quad \text{if } i = j$$
$$= 0 \quad \text{if } i \neq j$$

Show that $\{l_1, \ldots, l_n\}$ is a basis for V': it is called the <u>dual</u> <u>basis</u>

(2) Find the basis $\{e_1, e_2, e_3\}$ in P_2 for which l_1, l_2, l_3 above is the dual basis in P_2'.

4. Let W_1 and W_2 be subspaces of a vector space V such that $H = W_1 \cup W_2$ is also a subspace. Show that one of the spaces W_i is contained in the other

5. (a) Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace

(b) Prove that the only subspaces of $\mathbb{R}^2$ are $\mathbb{R}^2$, the zero subspace, or scalar multiples of some fixed vector in $\mathbb{R}^2$

(c) Describe all the subspaces of $\mathbb{R}^3$.

6. Let V be the set of real numbers. Regard V as a vector space over the field of <u>rational</u> numbers, with the usual operations. Prove that the vector space is <u>not</u> finite dimensional.