Linear Algebra
Problem set #7
Due: Nov 8, 2018

November 4, 2018

1. Use Jordan form to show that the matrices with distinct eigenvalues are dense in the space of all matrices.

2. Let A be the complex 3×3 matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{pmatrix}$$

Compute the Jordan form for A and show that A is similar to a diagonal matrix if and only if $a=0$

3. Let $a_0, a_1, \ldots, a_{n-1}$ be complex numbers and let V be the space of all n time's differentiable functions f on an interval of the real line which satisfy the differentiable equation

$$D^n f + a_{n-1} D^{n-1} f + \cdots + a_1 Df + a_0 f = 0$$

where D is the differential operator d/dx. Clearly V is invariant under D. What is the Jordan form for D on V?

Hints: Prove and use the following facts:

(a) By the Jordan form theorem every matrix A is similar to a matrix of the form

$$(1) \qquad \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}$$

where k is the number of distinct zeros of the characteristic polynomials for A, $p(s) = \det(s - A) = \prod_{j=1}^{k} (s - c_j)^{l_j}$, $l_j \geq 1$.

For each j, B_j is the generalized eigenspace for A corresponding to c_j and has the for

$$\begin{pmatrix} J_{j1} & & & \\ & J_{j2} & & \\ & & \ddots & \\ & & & J_{j,q_j} \end{pmatrix}$$

where each J_{jk}, $1 \leq k \leq q_j$ is an elementary Jordan block

$$\begin{pmatrix} c_j & 1 & & 0 \\ 0 & c_j & 1 & \\ & & \ddots & 1 \\ & 0 & & c_j \end{pmatrix}$$

such that $\dim J_{jk}$ is decreasing with k, but not strictly decreasing, from d_j = index of c_j to 1. <u>Show that</u> for each c_j

$$q_j = \text{dimension of the null space of } A - c_j$$

Show that $q_j = 1$ for all j, for $A - c = D - c$ in V.

(b) Show that the minimum polynomial $m_A(s)$ for A divides $P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$

(c) Let $P(s) = \prod_{j=1}^{k} (s - c_j)^{r_j}$

Show that $m_A(s) = P(s)$ and that A has block form as in (1) with each B_j equal to an elementary Jordan block

$$\begin{pmatrix} c_j & 1 & & 0 \\ 0 & c_j & 1 & \\ & & \ddots & 1 \\ 0 & & & c_j \end{pmatrix}$$

of dimension $d_j = r_j$

4. Let

$$X_1 \subset X_2 \subset \cdots \subset X_{k-1} \subset X_k$$

be a tower of linear spaces. Show that

$$\dim(X_1) + \dim(X_2/X_1) + \cdots + \dim X_k/X_{k-1} = \dim X_k$$