Linear Algebra
Problem Set #10
Due December 6, 2018

December 9, 2018

1. This problem provides a dynamical proof of the spectral theorem (Th^m 4' in Lax) for real self-adjoint matrices M_0. There exists a diagonal real matrix D and an orthogonal matrix Q, $Q^TQ = QQ^T = I$, such that $M_0 = QDQ^T$.

Proof: Let $M_0 = \bar{M}_0 = M_0^T$ be a given $n \times n$ matrix. Consider the differential equation for $M = M(t)$

(1) $$\begin{cases} \dfrac{dM}{dt} = [M, B(M)] = MB(M) - B(M)M \\ M(t=0) = M_0 \end{cases}$$

where $B(M) = M_- - M_-^T$ where M_- is the strictly lower triangular part of M. Thus $B = B(M)$ is skew-symmetric, $B = -B^T$, and

(2) $$\begin{aligned} (B(M))_{ij} &= M_{ij} && \text{if } i > j \\ &= 0 && \text{if } i = j \\ &= -M_{ij} && \text{if } i < j \end{aligned}$$

Equation (1) is called the Toda equation.

(i) Show that, by standard ODE techniques, (1) has a unique local solution $M(t)$, $0 \le t \le T$, for some $T > 0$,

where $M(0) = M$.

(ii) Show that

(3) $$\operatorname{tr} M^2 = \operatorname{tr}(M(t))^2$$

is constant and conclude that (1) has a unique global solution

$M(t)$, $0 \le t < \infty$, $M(0) = M$.

(iii) Let $M(t)$ be the solution of (1) and let $B = B(M(t))$. Show that the equation

(4) $$\frac{dQ}{dt} = QB$$

has a unique global solution $Q = Q(t)$, $t \ge 0$, $Q(0) = I$.

(iv) Show that Q is orthogonal for all $t \ge 0$,

(5) $$QQ^T = Q^TQ = I$$

and

(6) $$M(t) = Q(t)^T M_0 Q(t)$$

Conclude from (6) that

(7) $$\operatorname{spec} M(t) = \operatorname{spec} M_0, \quad t \ge 0$$

i.e. $t \mapsto M(t)$ is <u>isospectral</u>.

(v) Show that $\dfrac{dM_{11}}{dt} = 2 \sum_{j=2}^{n} M_{1j}^2 \ge 0$

and conclude using (3)

$$\sum_{i,j} M_{ij}^2(t) = \text{const}$$

that

(8) $$M_{11}(\infty) = \lim_{t\to\infty} M_{11}(t)$$

exists and

(9) $$M_{11}(\infty) = M_{11}(0) + 2\int_0^\infty \sum_{j=2}^n M_{1j}^2(t)\, dt$$

By (1) and (3) it follows that $\frac{dM}{dt}(t)$ is bounded for all $t \geq 0$. Conclude from (9) that

(10) $$\sum_{j=2}^n M_{1j}^2(t) \to 0 \quad \text{as } t \to \infty$$

(vi) Show similarly that $\frac{d}{dt}(M_{11} + \cdots + M_{kk}) \geq 0$ and conclude that for any $1 \leq k \leq n$

(11) $$\lim_{t\to\infty} M_{kk}(t) = M_{kk}(\infty)$$

exists and

(12) $$\sum_{j>k} M_{kj}^2(t) \to 0$$

Thus $\lim_{t\to\infty} M(t) = D$ exists with D diagonal.

(vii) Finally as the orthogonal matrices form a compact set, there exists $t_n \to \infty$ such that $Q(t_n) \to Q(\infty)$ for some orthogonal matrix $Q(\infty)$. But from (6), we have

$$M(t_n) = Q(t_n)^T M_0 \, Q(t_n)$$

and letting $t_n \to \infty$, ~~we~~ conclude that

(13) $$D = Q(\infty)^T M_0 \, Q(\infty)$$

or

$$M_0 = Q(\infty) \, D \, Q(\infty)^T$$

which proves the spectral theorem. Necessarily the diagonal elements of D are the eigenvalues of M_0.

2. Give a dynamical proof of the spectral theorem for (complex) self-adjoint matrices $M_0 = M_0^*$, i.e. there exists a real diagonal matrix D and a unitary matrix Q, $Q^*Q = QQ^* = I$ such that $M_0 = Q D Q^*$.

(Hint: Consider the differential equation

$$\frac{dM}{dt} = [M, B(M)], \quad t \geq 0, \quad M(0) = M_0$$

where $B(M) = M_- - M_-^*$.)

3. Use the Toda flow in (1) to prove the Wielandt-Hoffman inequality, viz., let M and N be real, symmetric matrices with eigenvalues m_j and n_j are both arranged in increasing, or decreasing order. Show that

(1) $$\sum_i (m_j - n_j)^2 \leq \operatorname{tr}(N-M)^2$$

Proof:

(i) Show that it is enough to prove that

(2) $$\sum m_j n_j \geq \operatorname{tr} NM$$

in the case that

(a) $\{m_j\}$ and $\{n_j\}$ are in decreasing order

(b) M is diagonal, i.e. $M = \operatorname{diag}(m_1, \dots, m_n)$

for (2) becomes

(3) $$\sum m_j n_j \geq \sum_{k=1}^{n} m_k n_{kk}$$

where $n_{kk} = N_{kk}$

(ii) Summing by parts, show that

(4) $$\sum_{k=1}^{n} m_k n_{kk} = m_n \sum_{k=1}^{n} n_{kk} + \sum_{k=1}^{n-1} (m_k - m_{k+1}) \left(\sum_{j=1}^{k} n_{jj} \right)$$

Now under the Toda flow (1) of Problem 1 applied to $N(t)$, with M fixed, $N(0)=N$,

$$\sum_{k=1}^{n} n_{kk} = \operatorname{tr} N(t) = \text{constant}$$

and

$$\sum_{j=1}^{k} n_{jj} = \sum_{j=1}^{k} n_{jj}(t) \quad \text{is increasing in time.}$$

Thus as $m_k \geq m_{k+1}$, it follows from (4) that $\sum_{k=1}^{n} m_k n_{kk}(t)$

is non-decreasing. Also as $t \to \infty$, $n_{kk}(t) \to d_k$,

where d_k, $k=1,\dots,n$ are the eigenvalues of N.

Thus $\sum m_k n_{kk}(t)$

$$\operatorname{tr} NM = \sum m_k n_{kk}(0) \leq \sum m_k n_{kk}(t) \leq \sum m_k d_k .$$

Now the d_k's are the eigenvalues of N, but they may be out of order. However

(iii) Show that if $m_1 \geq \dots \geq m_n$, and ~~then if~~ $d_1, \dots d_n$ is any other set of numbers, then

$$\sum_{k=1}^{n} m_k d_k \leq \sum_{k=1}^{n} m_k \hat{d}_k$$

where $\hat{d}_1 \geq \dots \geq \hat{d}_n$ is a monotonically decreasing reordering of $d_1, \dots, d_n$. This proves (3) and hence the Wielandt-Hoffman

automatically decreasing
i.e. Toda is a sorting algorithm

inequality.

(Note (i) the same result is true if X_T, N are complex Hermitian matrices. Prove it using Problem 2.)

(ii) actually one can show that generically the d_n's in the above proof, are $\wedge$.

4. Suppose T is a real tridiagonal matrix

$$T_{ij} = 0 \quad \text{if} \quad |i-j| > 1.$$

such that $T_{i\,i+1} \neq 0$, $i = 1, \dots, n-1$.

(a) Show it each eigenvalue of T is geometrically simple, i.e., $\dim \operatorname{Nul}(T-\lambda) = 1$ if λ is an eigenvalue of T

(b) If $u = (u_1, \dots, u_n)^T$ is an eigenvector for T, then $u_1 \neq 0$.

(c) Show that if T is real symmetric tridiagonal matrix with non-zero off-diagonal entries, then T has simple spectrum and if $u = (u_1, \dots, u_n)^T$ is an eigenvector, then $u_1 \neq 0$