Linear Algebra
Problem Set #8
Due Nov 29

November 22, 2018

1. An inner product (or scalar product) on a complex linear space X is a mapping from $X \times X \to \mathbb{C}$, denoted (x, y) for $x, y \in X$, with the following properties:

(i) $(x+y, z) = (x, z) + (y, z)$ for $x, y, z \in X$
(ii) $(\lambda x, y) = \lambda (x, y)$ for $x, y \in X$, $\lambda \in \mathbb{C}$
(iii) $(x, y) = \overline{(y, x)}$
(iv) $(x, x) > 0$ for $x \in X$, $x \neq 0$

Define $\|x\| = \sqrt{(x, x)}$, $x \in X$

Prove the following analogs for complex X of results for the real case:

(1) $|(x, y)| \leq \|x\| \, \|y\|$ $\forall \, x, y \in X$

(2) $\|x+y\| \leq \|x\| + \|y\|$ $\forall \, x, y \in X$

Note: For $X = \mathbb{C}^n$, the standard inner product is

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i \quad \text{for } x = (x_1, \ldots, x_n), \; y = (y_1, \ldots, y_n)$$

2. Let X be the space of polynomials on the interval $[-1, 1]$ with real coefficients and of degree $\le n$. X carries the natural inner product

$$(p, q) = \int_{-1}^{1} p(x)\, q(x)\, dx \quad , \quad p, q \in X$$

For $l = 0, 1, 2, \ldots$ define the polynomial of degree l

$$P_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

The P_l's are called <u>Legendre</u> polynomials. Set

$$q_l(x) \equiv \sqrt{\frac{2l+1}{2}}\; P_l(x)$$

(1) Show that q_l, $l = 0, \ldots, n$, is an orthonormal basis for X

(2) Show that $P_l(x)$ has the expansion

$$P_l(x) = \frac{1}{2^l} \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \binom{l}{k} \binom{2l-2k}{l} x^{l-2k}$$

where $\lfloor \cdot \rfloor$ is the floor function, where $\lfloor c \rfloor$ is the largest integer $\le c$.

(3) Show that q_ℓ, $\ell = 0, \dots, n$ is also an orthonormal basis for the space of polynomials of degree $\leq n$ with complex coefficients and inner product on $[-1, 1]$

$$\int_{-1}^{1} p(x)\,\overline{q(x)}\,dx$$

3. For $-1 \leq x \leq 1$, set

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, \dots$$

(i) Show that $T_n(x)$ is a polynomial of degree $\leq n$ for $n = 0, 1, \dots$ The T_n's are called Chebyshev polynomials

(ii) Let X be the space of real polynomials of degree $\leq N$ on $(-1, 1)$ with inner product

$$(p, q) \equiv \int_{-1}^{1} p(x)\,q(x)\,\frac{dx}{\sqrt{1-x^2}}$$

Let
$$t_n = \sqrt{\frac{2}{\pi}}\,T_n, \quad n > 0$$
$$= \frac{1}{\sqrt{\pi}}\,T_0, \quad n = 0$$

Show that $t_0, \dots, t_N$ is an orthonormal basis for X.

4. In an inner product space $(X, (\cdot,\cdot))$ we defined the norm (or length) of x by

(0) $$\|x\| = \sqrt{(x, x)}$$

and showed that

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$

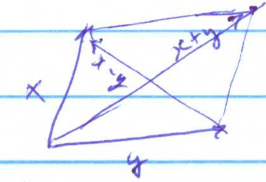
(ii) $\|\alpha x\| = |\alpha|\,\|x\|$, $\alpha \in \mathbb{R}$, $x \in X$

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in X$

Any mapping $\|\cdot\| : X \to [0, \infty)$ with properties (i) (ii) (iii) is called a norm and $(X, \|\cdot\|)$ is called a normed linear space. Not all norms arise from an inner product as in (0).

(a) Show that if $\|\cdot\|$ arises from an inner product as in (0), then the parallelogram law holds

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$



(b) Conversely show that if the parallelogram law holds for $(X, \|x\|)$, then

$$(x, y) \equiv \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

defines an inner product on X and $\|x\| = \sqrt{(x, x)}$.

(c) For $X = \mathbb{R}^n$, set

$$\|x\|_m \equiv \max_i |x_i|, \qquad x = (x_1, \dots, x_n)$$

Show that $\|x\|_m$ is a norm on X, which does not arise from an inner product on X

(d) Let X be the linear space of continuous functions on $[0,1]$ and set

$$\|f\|_c \equiv \int_0^1 |f(t)|\,dt$$

Show that $\|f\|_c$ is a norm on X, which does not arise from an inner product.

5. Let $(X, \|\cdot\|)$ be a finite dimensional normed space as in Problem 4, $\dim X = n < \infty$. For $\ell \in X'$, the dual space of X, define

(*) $$\|\ell\| \equiv \sup_{\|x\| \leq 1} |\ell(x)|$$

(a) Show that $\|\ell\|$ is a norm on X'

(b) Show that there is a basis $x_1, \ldots, x_n$ for X and a basis $\ell_1, \ldots, \ell_n$ for X' such that

- $\|x_i\| = 1$, $i = 1, \ldots, n$
- $\|\ell_j\| = 1$, $j = 1, \ldots, n$
- $\ell_j(x_i) = \delta_{ij}$, $1 \leq i, j \leq n$

(Hint: Consider $H(y_1, \ldots, y_n) = \det(y_1, \ldots, y_n)$ for $y_1, \ldots, y_n \in X$.)

6. Let $(X, \|\cdot\|)$ be a normed linear space and let A be a linear mapping from X to X. In analogy to (*) above, set $\|A\| \equiv \sup\{\|Ax\| : x \in X, \|x\| \leq 1\}$

(a) Show that $\|A\|$ is a norm on $\mathcal{L}(X)$, the linear mappings from $X \to X$.

(b) Let $(X, \|\cdot\|)$ be finite dimensional linear space, $\dim X = n$, and Y be a subspace of X. Show that there is a projection P_Y of X onto Y such that

$$\|P_Y\| \le n$$

(Hint: use problem 5(b)).

7. Find the adjoint of the differential operator

$$L(f)(x) = \frac{df(x)}{dx}$$

in the space of smooth periodic functions in $(0, 2\pi)$ with inner product

$$(f, g) = \int_0^{2\pi} f(x)\, g(x)\, (2 + \sin x)\, dx .$$