Solutions for Problem Set #9 LA 2018

taken from solution sets of students

Q1.

(a) $a_{m+n} = \log \|A^{m+n}\| = \log \|A^m \cdot A^n\| \le \log \|A^m\| \|A^n\|$

$= \log \|A^m\| + \log \|A^n\| = a_m + a_n$ $\qquad \|CA\| \le \|C\| \|A\|$

$\Rightarrow a_{m+n} \le a_m + a_n$

(b) $a_n = a_{mq+r} \le a_{mq} + a_r \Rightarrow \frac{a_n}{n} \le \frac{a_{mq} + a_r}{n}$

$\frac{a_n}{n} \le \frac{a_{mq} + a_r}{mq + r}$ $\qquad a_{k+k} \le a_k + a_k = 2a_k$

$\Rightarrow$ by induction $a_{mq} \le q \cdot a_m$ ✓

$\frac{a_n}{n} \le \frac{q a_m + a_r}{mq + r} = \frac{a_m + \frac{a_r}{q}}{m + \frac{r}{q}} \le \frac{a_m + \frac{a_r}{q}}{m}$

$\overline{\lim_n} \frac{a_n}{n} \le \overline{\lim_n} \frac{a_m + \frac{a_r}{q}}{m} = \frac{a_m}{m} + \overline{\lim_n} \frac{a_r}{mq}$ ✓

$a_r \le r a_1 \Rightarrow \frac{a_r}{mq} \le \frac{r}{mq} a_1$ nice. as $n \to \infty$ $q \to \infty$

but $r \le m-1 \Rightarrow \frac{a_r}{mq} \le \underbrace{\frac{(m-1)}{m} a_1}_{< \infty} \cdot \underbrace{\frac{1}{q}}_{\to 0}$ ✓

$\Rightarrow \overline{\lim_n} \frac{a_r}{mq} = 0 \Rightarrow \overline{\lim_n} \frac{a_n}{n} \le \frac{a_m}{m}$

(c) $\overline{\lim_{n \to \infty}} \frac{a_n}{n} \le \frac{a_m}{m} \quad \forall m \Rightarrow \overline{\lim_{n \to \infty}} \frac{a_n}{n} \le \inf_m \frac{a_m}{m} \le \underline{\lim_n} \frac{a_m}{m}$

$\Rightarrow \overline{\lim_n} \frac{a_n}{n} \le \underline{\lim_n} \frac{a_n}{n} \le \overline{\lim} \frac{a_n}{n} \Rightarrow \overline{\lim_n} \frac{a_n}{n} = \underline{\lim} \frac{a_n}{n}$ ✓

$$\Rightarrow \exists \lim_{n\to\infty} \frac{a_n}{n} \qquad \overline{\lim_{n}} \frac{a_n}{n} \leq \underline{\inf \frac{a_n}{n}} \leq \underline{\lim_{n}} \frac{a_n}{n} \quad \Rightarrow \lim_{n\to\infty} \frac{a_n}{n} = \inf \frac{a_n}{n}$$

Problem 2.

Solution: Note that for any $f \in X$, we have

$\|Tf\| = \| \int_0^x f(t)\,dt \| \leq \| \int_0^1 \sup|f(t)|\,dt \| \leq \sup|f(t)| = \|f\|$

Therefore, we conclude that $\|T\| \leq 1$.

Consider the special case $f \equiv 1$. then $Tf = x$

Thus we have $\|Tf\| = \sup_{[0,1]} |x| = 1$

Hence we have shown that $\|T\| = 1$.

Next we will consider $\|T^n\|$.

Note that we have

$$\begin{aligned} \|T^2 f\| &= \| \int_0^x \int_0^y f(t)\,dt\,dy \| \\ &\leq \| \int_0^1 \int_0^y \|f\|\,dt\,dy \| \\ &\leq \| \int_0^1 y\|f\|\,dy \| = \frac{1}{2}\|f\| \end{aligned}$$

This shows that $\|T^2\| \leq \frac{1}{2}$. Take $f \equiv 1$, we can show that $\|T^2\| = 1$.

By a completoly same method of calculation, we can conclude that $\|T^n\| = \frac{1}{n!}$

Note that we have the relation $r(T) \leq \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$

By Stirling's formula, we have that $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$.

This shows that $0 \leq r(T) \leq \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} \Longrightarrow r(T) = 0$.

Thus we have shown that $r(T) = 0$ and $\|T\| = 1$.

Problem 3

Solution: Consider any $x = \lambda_1 u + \mu_1 v$ and $y = \lambda_2 u + \mu_2 v$.

Then we have the following:

$$\langle Px, y\rangle = \langle \lambda_1 u, \lambda_2 u + \mu_2 v\rangle$$
$$= \lambda_1\lambda_2 \langle u,u\rangle + \lambda_1 \mu_2 \langle u,v\rangle.$$

On the other hand, we have: (Suppose $P^* y = au + bv$).

$$\langle x, P^* y\rangle = \langle \lambda_1 u + \mu_1 v, au + bv\rangle$$
$$= \lambda_1 a\langle u,u\rangle + \lambda_1 b\langle u,v\rangle + \mu_1 a\langle u,v\rangle + \mu_1 b\langle v,v\rangle.$$

Since $\langle Px, y\rangle = \langle x, P^* y\rangle$, we obtain the equation:

$$\lambda_1\left[\lambda_2\langle u,u\rangle + \mu_2\langle u,v\rangle - a\langle u,u\rangle - b\langle u,v\rangle\right]$$
$$-\mu_1\left[a\langle u,v\rangle + b\langle v,v\rangle\right] = 0.$$

This holds for any $x = \lambda_1 u + \mu_1 v$, therefore we have:

$$\begin{cases} a\langle u,u\rangle + b\langle u,v\rangle = \lambda_2\langle u,u\rangle + \mu_2\langle u,v\rangle \\ a\langle u,v\rangle + b\langle v,v\rangle = 0. \end{cases}$$

Solve the equation we have

$$a = \frac{\langle u,u\rangle\langle v,v\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\lambda_2 + \frac{\langle u,v\rangle\langle v,v\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\mu_2$$

$$b = -\frac{\langle u,u\rangle\langle u,v\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\lambda_2 - \frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\mu_2$$

Thus, we conclude that

$$P^*(y) = P^*(\lambda_2 u + \mu_2 v)$$
$$= \left[\frac{\langle u,u\rangle\langle v,v\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\lambda_2 + \frac{\langle u,v\rangle\langle v,v\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\mu_2\right]u$$
$$- \left[\frac{\langle u,u\rangle\langle u,v\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\lambda_2 + \frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\mu_2\right]v$$

We claim that P^* is ~~still~~ also some projection.

To see this, we have that

$$P^* = (P^2)^* = P^* P^* = (P^*)^2$$

Thus, we conclude that P^* is also a projection.

More precisely, if we take $y = t(\langle u, v\rangle u - \langle u, u\rangle v)$, $t \in \mathbb{R}$

then we will have $P^*(y) = 0$

Let $W = \{t(\langle u, v\rangle u - \langle u, u\rangle v) : t \in \mathbb{R}\}$

And if we take $y = t(-\langle v, v\rangle u + \langle u, v\rangle v)$, $t \in \mathbb{R}$

then we will have $P^* y = y$.

Let $Y = \{t(-\langle v, v\rangle u + \langle u, v\rangle v) : t \in \mathbb{R}\}$.

Then we conclude that P^* is the projection of $\mathbb{R}^2$ onto Y along W.

where $Y = \{t(-\langle v, v\rangle u + \langle u, v\rangle v) : t \in \mathbb{R}\}$

$$W = \{t(\langle u, v\rangle u - \langle u, u\rangle v) : t \in \mathbb{R}\}.$$

Problem 4.

$$q(y) = \sum_{i=1}^{3}(2+v_i)y_i^2 - 2\sum_{i=1}^{3} y_i y_{i+1} = (2+v_1)(y_1 - \frac{y_2+y_3}{2+v_1})^2 + R_1(y_2,y_3)$$

where

$$R_1(y_2,y_3) = -\frac{(y_2+y_3)^2}{2+v_1} + (2+v_2)y_2^2 + (2+v_3)y_3^2 - 2y_2y_3$$

Setting $d_1 = 2+v_1$ and $z_1 = y_1 - (y_2+y_3)/(2+v_1)$, we get

$$q(y) = d_1 z_1^2 + R_1(y_2,y_3)$$

Continuing with $R_1(y_2,y_3)$, compute

$$R_1(y_2,y_3) = (2+v_2 - \frac{1}{d_1})y_2^2 - 2(\frac{y_3}{d_1} + y_3)y_2 - \frac{y_3^2}{d_1} + (2+v_3)y_3^2$$

Setting $d_2 = 2+v_2 - 1/d_1$ and $z_2 = y_2 - y_3(1/d_2 + 1/(2d_2 + v_1 d_2))$, we get

$$R_1(y_2,y_3) = d_2 z_2^2 + R_2(y_3)$$

Evaluate $R_2(y_3)$ to get d_3: Calculation here is generic, eg. $2+v_1 \neq 0$.
what happens in the non-generic cases?

Problem 5.

...tion: (a) For the matrix $H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$, we have

$$q(y) = y_1^2 - 4y_2^2 + 6y_3^2 + 4y_1y_2 + 10y_2y_3 + 6y_3y_1$$

$$= \left(y_1^2 + 2y_1(2y_2+3y_3) + (2y_2+3y_3)^2\right) - (2y_2+3y_3)^2$$
$$-4y_2^2 + 6y_3^2 + 10y_2y_3$$

$$= [y_1+2y_2+3y_3]^2 - 4y_2^2 - 9y_3^2 - 12y_2y_3 - 4y_2^2 + 6y_3^2 + 10y_2y_3.$$

$$= [y_1+2y_2+3y_3]^2 - 8y_2^2 - 3y_3^2 - 2y_2y_3.$$

$$= [y_1+2y_2+3y_3]^2 - 8\left[y_2^2 + 2\cdot y_2\cdot\tfrac{1}{8}y_3 + \tfrac{1}{64}y_3^2\right] + \tfrac{1}{8}y_3^2 - 3y_3^2$$

$$= [y_1+2y_2+3y_3]^2 - 8\left[y_2 + \tfrac{1}{8}y_3\right]^2 - \frac{23}{8}y_3^2$$

Let $\begin{cases} z_1 = y_1 + 2y_2 + 3y_3 \\ z_2 = y_2 + \frac{1}{8}y_3 \\ z_3 = y_3 \end{cases}$ then we have $q(y) = z_1^2 - 8z_2^2 - \frac{23}{8}z_3^2$

The linear transfrm L is $z = Ly$:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{8} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$d_1 = 1 \quad d_2 = -8 \quad d_3 = -\frac{23}{8}$ ✓

h) $K = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 0 \end{pmatrix} \rightarrow q = y_1 y_2 + \frac{5}{2} y_2 y_3 + \frac{3}{2} y_1 y_3$

$u_1 = \frac{1}{2}(y_1 + y_2) \quad u_2 = \frac{1}{2}(y_1 - y_2) \quad u_3 = y_3$

$\Rightarrow y_1 = u_1 + u_2, \quad y_2 = u_1 - u_2 \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = T^*$

$\Rightarrow q(y) = (y, Ky) = \left(\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} u, K \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} u \right) = \left(u, \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} u \right)$

$TKT = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 8 \\ -2 & 2 & -2 \\ 3 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 8 \\ 0 & -4 & -2 \\ 8 & -2 & 0 \end{pmatrix}$

$u_1 \quad u_2$

$\Rightarrow q(y) = 4y_1^2 - 4y_2^2 + 4y_1 y_3 - y_2 y_3$

$= 4\underbrace{(y_1 + \frac{1}{2} y_3)}_{=z_1}^2 - y_3^2 - 4y_2^2 - y_2 y_3$

$= 4z_1^2 - 4\underbrace{(y_2 - \frac{1}{8} y_3)}_{=z_2}^2 - \frac{1}{64} y_3^2 - y_3^2$

$=$ $\qquad z_3 = y_3$

$\Rightarrow q = 4z_1^2 - 4z_2^2 - \frac{65}{64} z_3^2$