Linear Algebra

Problem Set #9

Due December 6, 2018

December 2, 2018

1. Let A be an $n \times n$ matrix. We showed that

$$(1) \qquad \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

exists by an indirect argument is, we showed that

$$r(A) \le \underline{\lim}_{n} \|A^n\|^{\frac{1}{n}} \le \overline{\lim}_{n} \|A^n\|^{\frac{1}{n}} \le r(A).$$

We can show that the limit in (1) exists directly using the following sub-additivity argument, which shows in addition that $\lim_n \|A^n\|^{1/n} = \inf_n \|A^n\|^{\frac{1}{n}}$

(a) Set $a_n = \log \|A^n\|$ and prove that $a_{m+n} \le a_m + a_n$

(b) For a fixed positive integer m set $n = mq + r$ where q and r are positive integers and $0 \le r \le m-1$ (Euclidean algorithm). Using (a), conclude that

$$\overline{\lim}_{n} \frac{a_n}{n} \le \frac{a_m}{m}$$

(c) Prove that $\lim_{n\to\infty} \frac{a_n}{n}$ $\exists$ and in fact equals $\inf_n \frac{a_n}{n}$.

2. Let T be the Volterra operator

$$Tf(x) = \int_0^x f(t)\,dt$$

acting in the linear space $X = \{f \text{ is continuos on } [0,1]\}$.

Show that the spectral radius $r(T) = 0$. Compute $\|T\|$.

3. Let u and v independent vectors in $\mathbb{R}^2$. Let P be the projection of $\mathbb{R}^2$ onto $U = \{\lambda u : \lambda \in \mathbb{R}\}$ along $V = \{\mu v : \mu \in \mathbb{R}\}$

ie if $x = \lambda u + \mu v$ for $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$. Then $Px = \lambda u$.

Compute P^* and interpret your result geometrically.

4. Consider the quadratic form $q(y) = \sum_{i=1}^{3} (y_{i+1} - y_i)^2 + \sum_{i=1}^{3} v_i y_i^2$, $v_i \in \mathbb{R}$, and $y_4 = y_1$. Express q in diagonal form, i.e.,

$$q(L^{-1}z) = d_1 z_1^2 + d_2 z_2^2 + d_3 d_3^2$$

for some d_1, d_2, d_3.

5. Let

(a) $H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ and consider the quadratic form $q_H(y) = (y, Hy)$. Find variables $z = Ly$ which diagonalize q i.e. $q(L^{-1}z) = d_1 z_1^2 + d_2 z_2^2 + d_3 z_3^2$

and compute d_1, d_2, d_3

(b) Let $K = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 0 \end{pmatrix}$. Diagonalize $q_K(y) = (y, Ky)$

as in part (a).