Linear Algebra
Problem set #4
Due: Oct 18, 2018

October 11, 2018

1. Show that if A is a square matrix
$$\det A = \det A^T$$
where A^T is the transpose of A.

2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ be a matrix over a field K and (c_1, c_2, c_3) be the vector in K^3 defined by

$$c_1 = \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}, \quad c_2 = \det\begin{pmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{pmatrix}, \quad c_3 = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Show that

(a) $\operatorname{rank}(A) = 2$ if and only if $(c_1, c_2, c_3) \neq (0,0,0)$

(b) If A has rank 2, then (c_1, c_2, c_3) is a basis for the solution space of the equation $Ax = 0$

3. Prove that for any real numbers $x_1, \dots, x_{n-1}$

$$\det\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ - & - & - & & - \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} = \prod_{n-1 \geq i > j \geq 1} (x_i - x_j)$$

This determinant is called the van der Monde determinant.

4. An $n \times n$ matrix is <u>triangular</u> if $A_{ij} = 0$ whenever $i > j$ (upper triangular) or $i < j$ (lower triangular). Show that $\det A = A_{11} A_{22} \cdots A_{nn}$ if A is triangular.

5. If A and B are $n \times n$ matrices, then $\det AB = \det A \det B$.

(a) If A is $n \times q$ and B is $q \times n$, with $q \geq n$, prove the Cauchy-Binet formula:

$$\det AB = \sum_{1 \leq i_1 < \cdots < i_n \leq q} \det \begin{pmatrix} a_{1 i_1} & \cdots & a_{1 i_n} \\ \vdots & & \vdots \\ a_{n i_1} & \cdots & a_{n i_n} \end{pmatrix} \det \begin{pmatrix} b_{i_1 1} & \cdots & b_{i_1 n} \\ \vdots & & \\ b_{i_n 1} & & b_{i_n n} \end{pmatrix}$$

where $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq q}}$ and $B = (b_{ij})_{\substack{1 \leq i \leq q \\ 1 \leq j \leq n}}$

(b) Use (a) to evaluate $\det AB$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{33} \end{pmatrix}$$

(c) If $q < n$ in (a), show $\det AB = 0$

6. Let A the $n \times n$ matrix $\begin{pmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ & & \vdots & & \\ a & a & \cdots & a & x \end{pmatrix}$

i.e. $A_{ij} = a$ if $i \neq j$, $A_{ii} = x$. Compute $\det A$

as a function of a and x.

7. Let A, B, C, D be square matrices of the same size.

If $AC = CA$, show that $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$.

(Hint: first consider the case where A is invertible.)