Linear Algebra
Problem set #5

Oct 18, 2018

Due: Oct 25, 2018

1. Let X be the linear space of lower triangular matrices, $A \in X$ iff $A_{ij} = 0$ for $j > i$. Show that the space of symmetric matrices $S = \{T : T_{ij} = T_{ji}\}$ is (isomorphic to) the dual space X' of X. (Hint: use tr)

2. Show that the equation $[A, B] = I$ has no solutions A, B in the space of $n \times n$ matrices. Here $[A, B]$ is the commutator $AB - BA$.

3. How many multiplications does it take to evaluate $\det A$ by using Gaussian elimination to bring it into upper triangular form? How many multiplications does it take to evaluate $\det A$ by the formula

$$\det A = \sum_p \sigma(p)\, a_{p_1 1} \cdots a_{p_n n} \; ?$$

4. Show that the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 1 & \cdot & & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & & \vdots \\ 1 & \cdots & & 1 & 0 \end{pmatrix}$$

i.e. $A_{ij} = 1$ for $i \neq j$
$= 0$ for $i = j$

has a complete set of eigenvectors. What are its eigenvalues? Compute $\det A$.

5. A Cauchy matrix is a matrix with entries

$$a_{ij} = \frac{1}{x_i - y_j}, \quad x_i - y_j \neq 0, \quad 1 \le i \le m, \ 1 \le j \le n,$$
$$x_i \neq x_j \text{ for } i \neq j, \quad y_i \neq y_j \text{ for } i \neq j.$$

Show that for $m = n$

$$\det A = \frac{\prod_{i=2}^{n} \prod_{j=1}^{i-1} (x_i - x_j)(y_j - y_i)}{\prod_{i=1}^{n} \prod_{j=1}^{n} (x_i - y_j)}$$

Conclude that a Cauchy matrix A is always invertible.

(Note: A Hilbert matrix is a special case of a Cauchy matrix where $x_i - y_j = i + j - 1$.)

6. Consider the $n \times n$ matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & \cdots & & a_{n-1} \\ \vdots & & & & \vdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_n & a_1 \end{pmatrix}$$

Show that the eigenvalues of A are of the form

$$\lambda_{k+1} = a_1 + a_2 w_k + a_3 w_k^2 + \cdots + a_n w_k^{n-1}, \quad 0 \le k \le n-1$$

where $w_k = e^{2\pi i k/n}$ are the nth roots of unity.

(Hint: the eigenvectors can be expressed in terms of the w_k's.)

(Note: A is called a circulant matrix.)