Lecture 12

We continue with the perturbation theory of multiple eigenvalues of $A(\beta)$, where $A(\beta)$ is analytic in some region $\Omega \subset \mathbb{C}$. Our goal now is to provide the analysis on which the various phenomena described in the previous lecture is based. From the examples of $A(\beta)$ in Lecture 9, one notices again the following interesting fact: both the matrices $\begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$ have a degeneracy (in this case a double eigenvalue) at $\beta = 0$. In the first case, the eigenvalues are $\pm\beta$, both of which are analytic at $\beta = 0$. In the second case, however the roots are $\pm\sqrt{\beta}$, which are <u>not</u> analytic at $\beta = 0$. The <u>fundamental difference</u> between these two examples, as we shall see, is that the first is Hermitian for real β, but the second is not.

The basic question in eigenvalue perturbation theory can clearly be rephrased in purely analytic terms in the following way:

What can we say about the roots λ of

$$F(\beta,\lambda) = \det(\lambda - A(\beta)) = \lambda^n + a_{n-1}(\beta)\lambda^{n-1} + \cdots + a_0$$

as β varies over Ω? Here $A(\beta)$ is $n\times n$.

So let us first consider more generally functions

(165.1) $$G(\beta,\lambda) = \lambda^n + g_{n-1}(\beta)\lambda^{n-1} + \cdots + g_0(\beta)$$

where $g_i(\beta)$ is analytic for $\beta \in \Omega$. The basic reference here is Knopp's book, Theory of Functions, II, Chap 5. We consider first the case where $G(\beta,\lambda)$ cannot be factored

(165.2) $$G(\beta,\lambda) = G_1(\beta,\lambda)\, G_2(\beta,\lambda)$$

where $G_i(\beta,\lambda)$, $i=1,2$, are of the same form as (165.1), with

$\deg_\lambda G_1(\beta,\lambda)\ \deg_\lambda G_2(\beta,\lambda) = n$. Such a situation arises,

for example, if

(165.3) $$A(\beta) = \left(\begin{array}{cc|cc} 0 & \beta & & \\ \beta & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & \beta \\ \multicolumn{2}{c|}{0} & \beta & 0 \end{array}\right)$$

Then $\det(\lambda - A(\beta)) = (\lambda^2-\beta^2)^2 = G_1(\beta,\lambda)\,G_2(\beta,\lambda)$ where $G_1(\beta,\lambda) = G_2(\beta,\lambda) = \lambda^2-\beta^2$. We see from this example that the factorization $G = G_1 G_2$ has something to do with $G(\beta,\lambda) = d(\lambda - A(\beta))$ having multiple roots

The following result is basic

Lemma 166.1

Suppose that $G(\beta,\lambda)$ has multiple roots for all $\beta \in \Omega$. Then G can be factorized $G = G_1 G_2$ as in (165.2).

Proof: If $G(\beta,\lambda)$ has a multiple root at λ, then

$$G(\beta,\lambda) = 0 \quad \text{and} \quad \frac{\partial G}{\partial \lambda}(\beta,\lambda) = 0$$

If $G(\beta,\lambda) = g_0(\beta) + \cdots + g_{n-1}(\beta)\lambda^{n-1} + \lambda^n$,

then

$$\frac{\partial G}{\partial \lambda}(\beta,\lambda) = g_1(\beta) \cdots + (n-1)g_{n-1}(\beta)\lambda^{n-2} + n\lambda^{n-1}$$

Now it is a basic algebraic result, that two λ-polynomials

$$p(\lambda) = p_0 + p_1\lambda + \cdots + p_n\lambda^n, \quad q(\lambda) = q_0 + q_1(\lambda) + \cdots + q_m\lambda^m$$

have a common root if and only if the resultant (or discriminant) $R(p,\lambda) = 0$, where $R(p,\lambda)$ is the $(m+n)\times(m+n)$ determinant

$$R(p,q) = \det \begin{array}{cc} \left(\begin{matrix} p_0 & p_1 & \cdots & p_n & 0 & \cdots & 0 \\ 0 & p_0 & \cdots & & p_n & & 0 \\ & & \ddots & & & & \\ & & & p_0 & \cdots & & p \\ q_0 & \cdot & q_1 & & q_m & 0 \cdots & 0 \\ 0 & q_0 & \cdots & & q_m & \cdots & 0 \\ & & q_0 & \cdots & & & q_m \end{matrix}\right) & \begin{matrix} \updownarrow m \\ \\ \\ \updownarrow n \end{matrix} \end{array}$$

For example if $p(\lambda) = p_0 + p_1\lambda + p_2\lambda^2$, $p_2 \neq 0$

and $q(\lambda) = p'(\lambda) = p_1 + 2p_2\lambda$,

then

$$R(p,q) = \det \begin{pmatrix} p_0 & p_1 & p_2 \\ p_1 & 2p_2 & 0 \\ 0 & p_1 & 2p_2 \end{pmatrix} \begin{array}{l} 1 \\ 2 \\ \end{array}$$

(167.1) $R(p,q) = p_0 4p_2^2 - p_1^2 p_2 = 0 \iff p_1^2 = 4p_0p_2$

Now if $q(\lambda) = 0$, then $\lambda = -\dfrac{p_1}{2p_2}$, and

$$p\left(-\frac{p_1}{2p_2}\right) = p_0 + p_1\left(-\frac{p_1}{2p_2}\right) + p_2\frac{p_1^2}{4p_2^2}$$

$$= p_0 - \frac{p_1^2}{2p_2} + \frac{p_1^2}{4p_2} = p_0 - \frac{p_1^2}{4p_2}$$

which is zero by (167.1).

Now in the case where

$p(\lambda) = G(\beta,\lambda)$

$q(\lambda) = \dfrac{\partial G}{\partial\lambda}(\beta,\lambda)$

$R(p,q)$ is clearly an analytic function of $\beta \in \Omega$.

So either (i) $R(p,q) = R(p(\beta), q(\beta)) \equiv 0 \quad \forall \beta \in \Omega$.

or (ii) $R(p,q) \neq 0$, except on a discrete set in Ω

Now (i) $\equiv$ $G(\beta,\lambda)$ has a multiple root for all $\beta \in \Omega$

To see how (i) $\Rightarrow$ $G = G_1 G_2$, consider the case $p = p_0 + p_1\lambda + \lambda^2$, $q = p_1 + 2\lambda$ about λ (with $p_2 = 1$), but now p_0, p_1 are analytic in some Ω_1. Then for each $\beta \in \Omega$, $\exists\ \mu = (\mu_1(\beta), \mu_2(\beta), \mu_3(\beta)) \neq 0$ such that

$$\mu_1 (p_0 \quad p_1 \quad 1) + \mu_2 (p_1 \quad 2 \quad 0) + \mu_3 (0 \quad p_2 \quad 2) = 0$$

which $\Rightarrow$ for any λ)

$$\mu_1 (p_0 + p_1\lambda + \lambda^2) + \mu_2 (p_1 + 2\lambda) + \mu_3 (p_1\lambda + 2\lambda^2)$$

Hence $\mu_1 p(\lambda) + (\mu_2 + \lambda\mu_3)(p_1 + 2\lambda) = 0$

Now if for some β, $\mu_1(\beta) = 0$, then $(\mu_2 + \lambda\mu_3)(p_1 + 2\lambda) = 0$ $\forall\lambda$, which is only possible if $\mu_2(\beta) = \mu_3(\beta) = 0$, which then contradicts $\mu \neq 0$. Thus $\mu_1(\beta) \neq 0$ in Ω.

But then $p(\lambda) = (s(\beta) + \lambda t(\beta))(p_1 + 2\lambda)$

where $s(\beta) = -\dfrac{\mu_2(\beta)}{\mu_1(\beta)}$, $t(\beta) = -\dfrac{\mu_3(\beta)}{\mu_1(\beta)}$ are analytic.

Thus $G = G_1 G_2$ in this case. The general case is left to the reader.

Note that if $R(p(\beta), q(\beta)) \neq 0$ for some β, condition (ii) then follows as $R(p(\beta), q(\beta))$ is analytic in β for

$$p(\beta) = \det(\lambda - A(\beta)), \quad q(\beta) = \frac{\partial}{\partial\lambda} \det(\lambda - A(\beta)). \quad \square$$

Corollary Let $A(\beta)$ be $n \times n$. If $G(\beta, \lambda) = \det(\lambda - A(\beta))$ cannot be factored as in (165.2) then $A(\beta)$ has algebraically simple roots $\lambda_1(\beta), \ldots, \lambda_n(\beta)$, $\lambda_i(\beta) \neq \lambda_j(\beta)$ for $i \neq j$, for all β away from a discrete set $D \subset \Omega$.

Illustrative example

Consider
$$A(\beta) = \begin{pmatrix} \beta & 1-\beta & 0 \\ 1-\beta & 0 & \beta \\ 0 & \beta & 1-\beta \end{pmatrix}$$

Have
$$\begin{aligned}
G(\beta, \lambda) &= \det(\lambda - A(\beta)) \\
&= (\lambda - \beta)\left(\lambda(\beta - 1 + \lambda) - \beta^2\right) - (\beta - 1)^2(\beta - 1 + \lambda) \\
&= \lambda^3 + \lambda^2(\beta - 1) - \lambda\beta^2 \\
&\quad - \beta\lambda^2 - \lambda\beta(\beta - 1) + \beta^3 \\
&\quad - (\beta - 1)^3 - (\beta - 1)^2\lambda \\
&= \lambda^3 - \lambda^2 + \lambda(-2\beta^2 + \beta - \beta^2 + 2\beta - 1) \\
&\qquad + \beta^3 + (1-\beta)^3 \\
&= (\lambda - 1)\lambda^2 - (\lambda - 1)(\beta^3 + (1-\beta)^3) \\
&= (\lambda - 1)\left(\lambda^2 - (\beta^3 + (1-\beta)^3)\right)
\end{aligned}$$

We have multiple roots when

(i) $\beta^3 + (1-\beta)^3 = 0$ or (ii) $\beta^3 + (1-\beta)^3 = 1$

(i) $1 - 3\beta + 3\beta^2 = 0 \iff \beta^2 - \beta + \frac{1}{3} = 0 \iff \beta = \dfrac{1 \pm \sqrt{1 - 4/3}}{2}$
$$= \frac{1 \pm i\,1/\sqrt{3}}{2}$$
$$= \frac{1}{2} \pm i\,\frac{1}{2\sqrt{3}}$$

(ii) $1-3\beta+3\beta^2=1 \iff \beta(\beta-1)=0 \iff \beta=0$ or 1

Thus $D = \left(\frac{1}{2} \pm \frac{i}{2\sqrt{3}}, 0, 1\right)$ so $G(\beta,\lambda)$ does not have multiple roots for all β. Nevertheless, $G(\beta,\lambda)$ can be factored

$$G(\beta,\lambda) = G_1(\beta,\lambda)\, G_2(\beta,\lambda)$$

where

$$G_1(\beta,\lambda) = \lambda - 1, \qquad G_2(\beta,\lambda) = \lambda^2 - (\beta^3 + (1-\beta)^3)$$

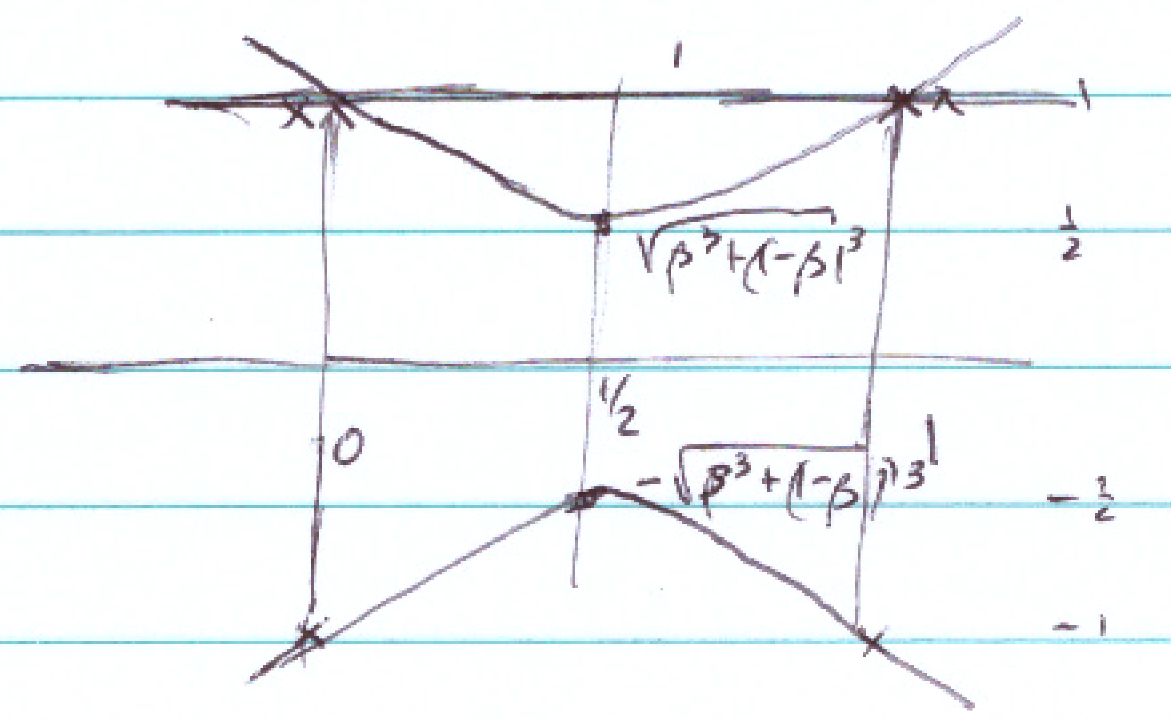
Thus the converse of Lemma 166.1 is not true in general.

Also note that the factorizability of $G(\beta,\lambda)$ as in (165.1), is a subtle business. The factorizability of $\det(\lambda - A(\beta))$ in (165.3) is obvious, but the factorizability of $\det(\lambda - A(\beta))$ in this example is not a priori obvious.

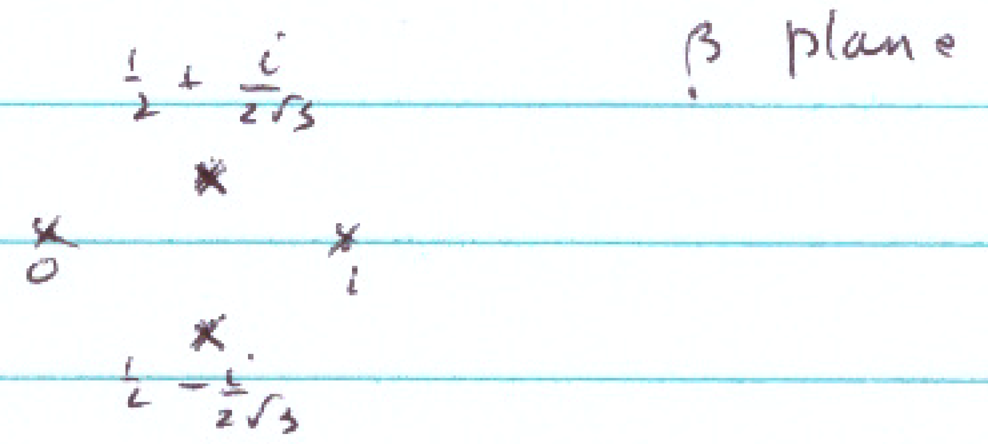
Now as $3\beta^2 - 3\beta + 1 = 3(\beta - \frac{1}{2})^2 + \frac{1}{4} \geq \frac{1}{4}$

the eigenvalues of $A(\beta)$ are 1, $\sqrt{\beta^3 + (1-\beta)^3}$, $-\sqrt{\beta^3+(1-\beta)^3}$

and for real β



and for D we have



The eigenvalues are analytic in β for β real, but there is branching at $\frac{1}{2} \pm \frac{i}{2\sqrt{3}}$.

Exercise Compute D for $A(\beta) = \begin{pmatrix} 1 & \beta \\ \beta & \beta^2 \end{pmatrix}$

Now suppose $G(\beta, \lambda) = g_0(\beta) + g_1(\beta)\lambda + \dots + g_{n-1}(\beta)\lambda^{n-1} + \lambda^n$ cannot be factored. Then as noted above $G(\beta,\lambda)$ has n simple roots $\lambda_1(\beta), \dots, \lambda_n(\beta)$ for β away from the discrete set D. Then as the roots are simple, $\frac{\partial G}{\partial \lambda}(\beta, \lambda_i(\beta)) \neq 0$ and so a standard implicit function theorem shows that each $\lambda_i(\beta)$ is analytic.

Knopp's main result is the following (see Knopp p121)

Theorem 171.1 Suppose $G(\beta,\lambda)$ as above cannot be factored. Then the n roots $\lambda_1(\beta), \dots, \lambda_n(\beta)$ of $G(\beta,\lambda)$ are the n values of a single n-valued analytic function $\lambda = F(\beta)$ for β in the punctured region, i.e. $\beta \in \Omega \setminus D$.

What this means is that there is one n-valued function $F(\beta) = \{F_1(\beta), \dots, F_n(\beta)\}$, and starting from any particular value, say $F_1(\beta)$ one obtains all the other values $F_2(\beta), \dots, F_n(\beta)$ by circling the points of D appropriately. For example, let

$$G(\beta,\lambda) = g_0(\beta) + g_1(\beta)\lambda + \lambda^2$$

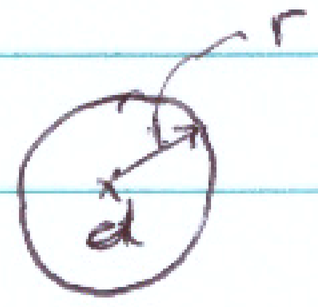
where $g_0(\beta) \not\equiv g_1^2(\beta)/4$ $\Leftrightarrow$ G cannot be factored

then the roots of $G(\beta,\lambda)=0$ are

$$\lambda_\pm(\beta) = \frac{-g_1(\beta) \pm \sqrt{g_1^2(\beta) - 4g_0(\beta)}}{2}.$$

Here $D = \{\beta : g_1^2(\beta) - 4g_0(\beta) = 0\}$

And for $d \in D$



, r small

starting with $\lambda_+(\beta)$, $\beta = d + r$
if we circle β around d by 2π, we obtain the other root, $\lambda_+(d + re^{2\pi i}) = \lambda_-(d)$

We need some preliminary results in order to prove Theorem 171.1. Fix $d \in D$. Relabeling if necessary, let p, $1 \le p \le n$, be the smallest number of rotations around d, (starting with $\lambda_1(\beta)$, $\beta \in \Omega \setminus D$,) until one returns to $\lambda_1(\beta)$. Thus

$$\lambda_1(\beta) \to \underset{=\lambda_2(\beta)}{\lambda_1(d+re^{2\pi i})} \to \underset{=\lambda_3(\beta)}{\lambda_1(d+re^{4\pi i})} \to \cdots \lambda_1(d+re^{2k\pi i}) = \lambda_1(\beta)$$

where $\lambda_1(d+re^{2(k-1)\pi i}) \ne \lambda_1(\beta)$ for $1 \le k < p$

Then $Q(\eta) = \lambda_1(\eta^p + d)$ is clearly a single valued analytic function in the punctured region $0 < |\eta| < r$

But $\lambda_1(\beta)$ is bounded near d. Hence $Q(\eta)$ extends to an analytic function in the full disk $\{|\eta| < r\}$, where it has a convergent power series.

$$Q(\eta) = \sum_{j=0}^{\infty} q_j \eta^j$$

Thus for β near d

(173.1) $$\lambda_1(\beta) = Q((\beta-d)^{\frac{1}{p}}) = \sum_{j=0}^{\infty} q_j (\beta-d)^{j/p}$$

Such a series is called a Puiseux series.

In order to prove Knopp's Theorem, note that it is enough to show that starting from fixed $\lambda_j(\beta)$, say $\lambda_1(\beta)$, one can obtain any other $\lambda_k(\beta)$, $k \ne 1$, by circling the points of D. If this is not true, then, relabeling if necessary, let $\lambda_1(\beta), \ldots, \lambda_k(\beta)$, $k < n$, be all the roots that can be obtained from $\lambda_1(\beta)$ by circling the points of D, and let $\lambda_{k+1}(\beta), \ldots, \lambda_n(\beta)$ be the roots of G that cannot be so obtained.

Let $G_1(\beta, \lambda) = \prod_{i=1}^{k}(\lambda - \lambda_i(\beta))$, $G_2(\beta,\lambda) = \prod_{i=k+1}^{n}(\lambda - \lambda_i(\beta))$

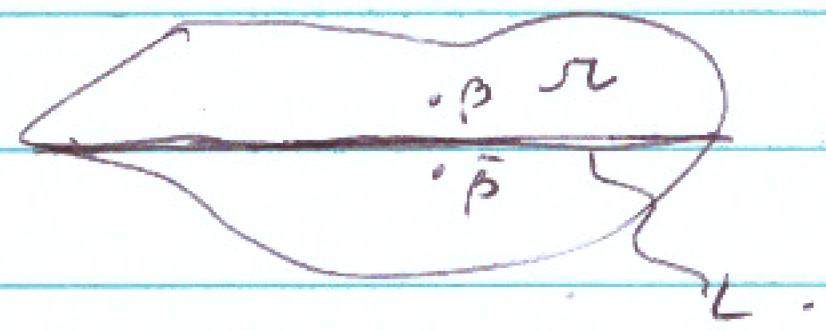
Then as $\lambda_1(\beta), \ldots, \lambda_k(\beta)$ are turned into each other after circling any $d \in D$, it follows that $G_1(\beta, \lambda)$ is single valued in $\Omega \setminus D$, and as it is bounded near each $d \in D$, it follows that $G_1(\beta, \lambda)$ is analytic in β and λ. The same is true for $G_2(\beta, \lambda)$, because any $\lambda_i(\beta)$, $i > k$, cannot turn into a $\lambda_j(\beta)$ for $j \le k$. We conclude that G can be factored, $G = G_1 G_2$, which is a contradiction.

This proves Knopp's Theorem 171.1 $\square$

We now focus on G of the form

(175.0) $$G(\beta,\lambda) = \det(\lambda - A(\beta))$$

where $A(\beta)$ is $n \times n$ and analytic for β in some region $\Omega \subset \mathbb{C}$. We say that the family $A(\beta)$, $\beta \in \Omega$, is self-adjoint if $A(\beta) = A(\beta)^*$ for $\beta \in \Omega \cap \mathbb{R} = L$



Necessarily $A(\beta) = A(\bar{\beta})^*$ for all β in Ω s.t. $\bar{\beta} \in \Omega$.

Theorem 175.1

Suppose $A(\beta)$ is a self-adjoint family, and $\lambda(\beta)$ is an eigenvalue of $A(\beta)$. Then $\lambda(\beta)$ is real analytic on $L = \Omega \cap \mathbb{R}$.

Proof: $G(\beta,\lambda)$ in (175.0) can be factored

(175.2) $$G = G_1 \cdots G_m$$

where each of the G_i's cannot be factored further. So we can assume that $\lambda(\beta)$ is a root of one of the G_i's, say G_1, $G_1(\beta, \lambda(\beta)) = 0$. Then for any real point $d \in D$

we have a (convergent) Puiseux series as in (173.1) for some $p \geq 1$.

$$\lambda_1(\beta) = \alpha_0 + \sum_{j=1}^{\infty} \alpha_j (\beta - d)^{j/p}$$

Now taking limits on the real axis, in particular,

$$\alpha_1 = \lim_{\beta \downarrow d} \frac{\lambda(\beta) - \alpha_0}{|\beta - d|^{1/p}}$$

as $(\beta - d)^{1/p} = |\beta - d|^{1/p}$ for $\beta > d$. But for β real, $A(\beta) = A(\beta)^*$

and so $\lambda_1(\beta)$ and $\alpha_0 = \lim_{\beta \downarrow 0} \lambda(\beta)$ are real. Hence α_1 is real.

But letting $\beta \uparrow d$,

$$\alpha_1 = \lim_{\beta \uparrow d} \frac{\lambda_1(\beta) - \alpha_0}{|\beta - d|^{1/p} e^{i\pi/p}}$$

and so $\alpha_1 e^{i\pi/p}$ is real. So if $p \neq 1$, then $\alpha_1 = 0$

By induction one shows that $\alpha_j = 0$ if j/p is not an integer. Therefore $\lambda(\beta)$ is actually analytic at $\beta = d$. $\square$

Taking into account that the same $\lambda(\beta)$ may arise from more than one factor G_i in (175.2), and using Rouche's theorem, we have proved the following theorem.

Theorem 177.1

Let $A(\beta)$ be an $n \times n$ matrix-valued analytic function in some region $\Omega \subset \mathbb{C}$. Then the eigenvalues of $A(\beta)$ are represented by s distinct functions

$\lambda_1(\beta), \quad \ldots, \lambda_s(\beta) \quad , \quad 1 \leq s \leq n$

These functions are continuous on Ω, analytic off a discrete set

$D \subset \Omega$ and have at worst algebraic singularities at

points of D. For β off D, $\lambda_i(\beta) \neq \lambda_j(\beta)$, $i \neq j$,

and at points $d \in D$, $\lambda_i(d) = \lambda_j(d)$ for some $i \neq j$.

The algebraic multiplicity $m_i(\beta)$ of $\lambda_i(\beta)$ is constant

for $\beta \in \Omega \setminus D$ and $\sum_{i=1}^{s} m_i(\beta) = n$, $\beta \in \Omega \setminus D$.

If $\lambda(\beta_0)$, $\beta_0 \in D$, is an eigenvalue of $A(\beta_0)$ of multiplicity

m_0, then, for β near β_0 $A(\beta)$ has exactly m_0

eigenvalues (counting multiplicity) near $\lambda(\beta_0)$. These eigenvalues are all branches of one or more multivalued functions analytic near β_0. If $A(\beta)$ is a self-adjoint family, then the

functions $\lambda_1(\beta), \ldots, \lambda_s(\beta)$ are real analytic on $L = \Omega \cap \mathbb{R}$. $\square$