Lecture 11 If A_0 is not self-adjoint, $A(\beta) = A_0 + \beta B$, the calculations for $\lambda(\beta)$ must be modified in the following way.

Again we consider an algebraically simple eigenvalue λ_0 of A_0, $A_0 v = \lambda_0 v$, and for $|\beta|$ small we have $P(\beta) = (u(\beta), \cdot)\, v(\beta)$ with $(u(\beta), v(\beta)) = 1$ and for $v(\beta) \equiv P(\beta) v_0$, $v(0) = v_0$

$$A(\beta)\, v(\beta) = \lambda(\beta)\, v(\beta)$$

where

(151.1) $$\lambda(\beta) = \frac{(v_0, A(\beta) v(\beta))}{(v_0, v(\beta))} \text{ and } v(\beta) \text{ are analytic in } \beta.$$

For $|\beta|$ small, $(v_0, v(\beta)) \neq 0$ so that for each k

$$(u(\beta), e_k) = \frac{(v_0, P(\beta) e_k)}{(v_0, v(\beta))}$$

is analytic. (Thus $\overline{u(\beta)} = f(\beta)$ where $f(\beta)$ is analytic, so $u(\bar\beta) = \overline{f(\bar\beta)}$ is analytic)

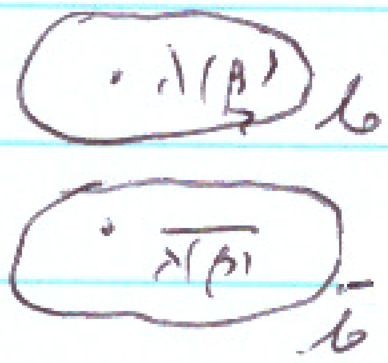
Now as $\lambda(\beta)$ is an eigenvalue of $A(\beta)$, $\overline{\lambda(\beta)}$ is an eigenvalue of $A(\beta)^*$. Moreover, as $\overline{\det(A-\lambda)} = \det(A^* - \bar\lambda)$, and as $\lambda(\beta)$ is algebraically simple we see that $\overline{\lambda(\beta)}$ must also be an algebraically simple eigenvalue of $A(\beta)^*$. The projection is again given by integration

$$P(A(\beta)^*) = \frac{1}{2\pi i} \int_{\tilde{\mathcal{C}}} \frac{dz}{z - (A(\beta))^*}$$

$\times\, \lambda(\beta)$

$\times\, \overline{\lambda(\beta)}$ $\tilde{\mathcal{C}}$

But it is clear that if $\mathcal{C}$ encircles $\lambda(\beta)$



then $\bar{\mathcal{C}}$ encircles $\overline{\lambda(\beta)}$, and

$$(P(H(\beta)))^* = \left(\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{dz}{z - H(\beta)}\right)^*$$

$$= -\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\overline{dz}}{\bar{z} - (H(\beta))^*}$$

$$= -\frac{1}{2\pi i}\oint_{\bar{\mathcal{C}}} \frac{dz}{z - H^*(\beta)}$$

$$= \frac{1}{2\pi i}\oint_{\bar{\mathcal{C}}} \frac{dz}{z - H^*(\beta)}$$

$$= P(H^*(\beta))$$

Thus

(152.1) $$P(H^*(\beta)) = (P(H(\beta)))^* = (v(\beta), \cdot\,)\,u(\beta)$$

from which it is clear that $u(\beta)$ is an eigenvector of $H^*(\beta)$ corresponding to $\overline{\lambda(\beta)}$, $H^*(\beta)\,u(\beta) = \overline{\lambda(\beta)}\,u(\beta)$.

Again as $(u(0), v(0)) = 1$, $(u(0), v(\beta)) \neq 0$ for β small.

We have

(153.1) $$\lambda(\beta) = (u_0, A(\beta)v(\beta)) / (u_0, v(\beta))$$
$$= \frac{(u_0, A(\beta)P(\beta)v)}{(u_0, P(\beta)v)}$$

This formula, as opposed to (151.1) is the one to use in the non-self-adjoint case, as we will soon see.

A similar calculation to the self-adjoint case (see p148, lecture 10) now gives

$$\lambda(\beta) = \frac{(u_0, A_0P_0v_0) + \beta\,\big((u_0, BP_0v_0) + (u_0, A_0P_1v_0)\big) + O(\beta^2)}{(u_0, P_0v_0) + \beta\,(u_0, P_1v_0) + O(\beta^2)}$$

$$= \frac{\lambda_0 + \beta\,\big((u_0, Bv_0) + \lambda(0)(u_0, P_1v_0)\big) + O(\beta^2)}{1 + \beta\,(u_0, P_1v_0) + O(\beta^2)}$$

as $P_0v_0 = v_0$, $A_0^*u_0 = \overline{\lambda(0)}\,u_0$. And again, as on p148, Lecture 10, $(u_0, P_1v_0) = \frac{1}{2\pi i}\oint_{|z-\lambda_0|=\varepsilon}\left(\frac{1}{z-A_0^*}u_0, B\frac{1}{z-A_0}v_0\right)dz$

$$= \frac{1}{2\pi i}\oint_{|z-\lambda_0|=\varepsilon}\left(\frac{1}{z-\bar\lambda_0}u_0, B\frac{1}{z-\lambda_0}u_0\right) = \frac{1}{2\pi i}\oint_{|z-\lambda_0|=\varepsilon}\frac{(u_0, Bu_0)\,dz}{(z-\lambda_0)^2}$$

$= 0$, as before.

Thus

(154.1) $$\lambda(\beta) = \lambda_0 + \beta\,(u_0, B\,v_0) + O(\beta^2)$$

The difference between the self adjoint case, $A_0 = A_0^*$, and the non-self-adjoint case, $A_0 \neq A_0^*$, is that the eigenvector u_0 of $A(0)^*$ appears in (154.1), as opposed to v_0 in (148.1). Of course if $A_0 = A_0^*$, then $u_0 = v_0$ and (154.1) reduces to (148.1). Again we have $v(\beta) = v_0 + \beta P_1 v_0 + O(\beta^2)$, but $P_1 v_0$ is no longer simple to evaluate, as in (149.1), as A_0 does not in general have a complete orthogonal set of e'vectors.

The above calculations using projections may appear a little cumbersome. Once one knows that the eigenvalues and the eigenvectors are analytic in β, $|\beta|$ small, one can simply write

$$\lambda(\beta) = \lambda_0 + \beta a_1 + \beta^2 a_2 + \cdots$$
$$v(\beta) = v_0 + \beta v_1 + \beta^2 v_2 + \cdots$$

and solve for the coefficients by substitution in the eigenvalue equation. Indeed

$$(A_0 + \beta B)(v_0 + \beta v_1 + \cdots) = (\lambda_0 + \beta a_1 + \cdots)(v_0 + \beta v_1 + \cdots)$$

$$\Rightarrow \quad A_0 v_0 = \lambda_0 v_0$$
$$B v_0 + A_0 v_1 = \lambda_0 v_1 + v_0$$
$$\vdots$$

Taking the inner product of the second equation with u_0, the eigenvector of A_0^*, $A_0^* u_0 = \bar{\lambda}_0 u_0$, we get

$$(u_0, Bv_0) + \lambda_0 (u_0, v_1) = \lambda_0 (u_0, v_1) + a_1 (u_0, v_0)$$

and so

$$a_1 = (u_0, Bv_0) \qquad (\text{recall } (u_0, v_0) = 1)$$

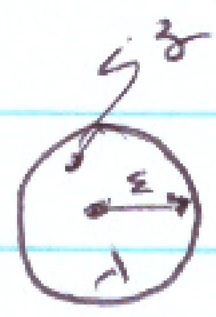
Thus

$$\lambda(\beta) = \lambda_0 + (u_0, Bv_0)\beta + O(\beta^2), \text{ as above.}$$

<u>But</u> remember the above calculation is predicated on the previously proved fact that $\lambda(\beta)$ and $v(\beta)$ are analytic.

Now suppose λ is an eigenvalue of a matrix A.

<u>Question</u>: How does $(z - A)^{-1}$ behave in a (small) neighborhood $N_\varepsilon(\lambda) = \{z : 0 < |z - \lambda| < \varepsilon\}$ of λ?



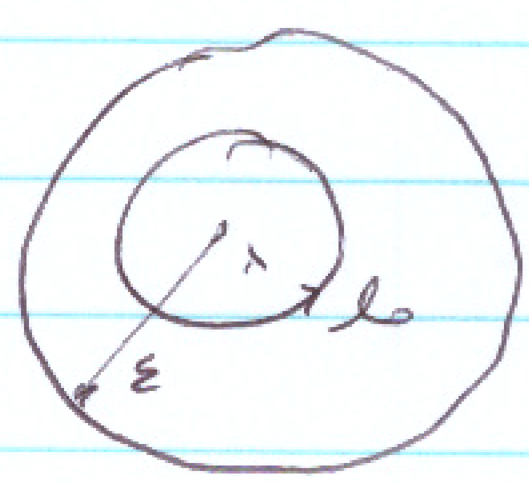
As $(z - A)^{-1}$ is analytic in $N_\varepsilon(\lambda)$, it has a convergent Laurent series expansion for $0 < |z - \lambda| < \varepsilon$,

(155.1) $$\frac{1}{z - A} = \sum_{n=0}^{\infty} A_n (z - \lambda)^n + \frac{C}{z - \lambda} + \sum_{n=1}^{\infty} \frac{B_n}{(z - \lambda)^{n+1}}$$

What are the matrices A_n, C and B_n?

Consider for any $m \in \mathbb{Z}$,

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(z - \lambda)^m}{\lambda - A} \, dz$$



$$= \sum_{n=0}^{\infty} A_n \int_{\mathcal{C}} (z-\lambda)^{m+n} \frac{dz}{2\pi i} + C \int_{\mathcal{C}} (z-\lambda)^{m-1} \frac{dz}{2\pi i}$$

$$+ \sum_{n=1}^{\infty} B_n \int_{\mathcal{C}} (z-\lambda)^{m-n-1} \frac{dz}{2\pi i}$$

If $m=0$, RHS $= C$. But $\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(z-\lambda)^0}{z-A} dz = P$ and so

(156.1) $$C = P$$

If $m \geq 1$, RHS $= B_m$, and so

(156.2) $$B_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(z-\lambda)^n dz}{z-A}, \quad n \geq 1$$

If $m \leq -1$, RHS $= A_{-m-1}$, and so

(156.3) $$A_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{1}{(z-\lambda)^{n+1}(z-A)} dz, \quad n \geq 0.$$

Now for $n \geq 1$,

$$B_n = \frac{1}{2\pi i} \int \frac{(z-\lambda)^n}{z-A} dz$$

$$= \frac{1}{2\pi i} \int (z-\lambda)^{n-1} \left(\frac{z-A+A-\lambda}{z-A} \right) dz$$

$$= (A-\lambda) B_{n-1}$$

Also $$B_1 = \frac{1}{2\pi i} \int (z-\lambda) \frac{dz}{z-A} = (A-\lambda)P$$

and so by induction

(156.4) $$B_n = (A-\lambda)^n P, \quad n \geq 1.$$

Similarly, if we set

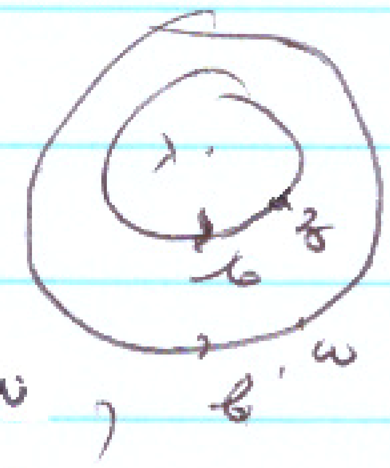
(156.5) $$S = A_0 = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{dz}{(z-\lambda)(z-A)}$$

we find

(157.1) $$A_n = (-1)^n S^{n+1}, \quad n \geq 0$$

Indeed, assume is true for n: the case $n=0$ is true by definition. Then by induction:

$$S^{n+2} = S\, S^{n+1} = (-1)^n S A_n$$



$$= \frac{(-1)^n}{(2\pi i)^2}\left(\int_{C} \frac{1}{z-\lambda}\,\frac{dz}{z-A}\right)\left(\int_{C'} \frac{1}{(w-\lambda)^{n+1}}\,\frac{dw}{w-A}\right)$$

$$= \frac{(-1)^n}{(2\pi i)^2}\int_{C}\int_{C'} \frac{dz\,dw}{w-z}\,\frac{1}{(z-\lambda)(w-\lambda)^{n+1}}\left(\frac{1}{z-A}-\frac{1}{w-A}\right)$$

$$= \frac{(-1)^n}{(2\pi i)^2}\int_{C} \frac{dz}{z-\lambda}\,\frac{1}{z-A}\int_{C'}\frac{dw}{(w-z)(w-\lambda)^{n+1}}$$

$$- \frac{(-1)^n}{(2\pi i)^2}\int_{C'}\frac{dw}{(w-\lambda)^{n+1}}\,\frac{1}{(w-A)}\int_{C} \frac{dz}{(w-z)(z-\lambda)}$$

Now $\int_{C'} \frac{dw}{w-z\,(w-\lambda)^{n+1}} = 0$, as we see by letting $C' \to \infty$. On the other hand $\frac{1}{2\pi i}\int_{C} \frac{dz}{(w-z)(z-\lambda)} = \frac{1}{w-\lambda}$

Hence $S^{n+2} = (-1)^{n+1} A_{n+1}$, which proves (157.1)

Now as noted before, by Cramer's rule $(z-A)^{-1}$ has a singularity at λ of order equal to the algebraic multiplicity of the eigenvalue λ. Thus for $n \geq$ algebraic mult.

of λ, $B_n = 0$, i.e.

(158.1) $$(A-\lambda)^n P = 0 \quad \text{for } n \geq \text{alg. mult. of } \lambda$$

In other words, as Ran P is a reducing for A, $(A-\lambda) \upharpoonright \text{Ran } P$ is nil-potent.

We see then that in a neighborhood of λ,

(158.2) $$\frac{1}{z-A} = \frac{(A-\lambda)^{m-1}P}{(z-\lambda)^m} + \cdots + \frac{(A-\lambda)P}{(z-\lambda)^2} + \frac{P}{z-\lambda}$$
$$+ \sum_{n=0}^{\infty} (-1)^n (z-\lambda)^n S^{n+1}$$

where $m \leq$ algeb. mult. of λ.

If A is diagonalizable at, then we can take $m=1$

i.e. $$\frac{1}{z-A} = \frac{P}{z-\lambda} + S - (z-\lambda)S^2 + \cdots$$

One can see this in two ways: in the reducing space Ran P, A has spectrum equal to the one point λ.

But A has a full set of vectors in Ran P and so

$$A \upharpoonright \text{Ran } P = U \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} U^{-1} = \lambda U U^{-1} = \lambda \quad \text{i.e. } (A-\lambda)P = 0$$

and so $(A-\lambda)^k P = 0 \;\; \forall\, k \geq 1$. Alternatively, if $A = V \Lambda V^{-1}$

$$B_n = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(z-\lambda)^n}{z-A}\,dz = V \begin{pmatrix} \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(z-\lambda)^n}{(z-\lambda_1)}\,dz & & 0 \\ & \ddots & \\ 0 & & \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(z-\lambda)^n}{z-\lambda_n}\,dz \end{pmatrix} V^{-1}$$

$\lambda = \lambda_1$, $\mathcal{C}$ $\lambda_2 \;\cdots\; \lambda_n$

$= 0$ if $n \geq 1$.

In particular if A is Hermitian, or normal, we see that $\frac{1}{z - A}$ has at worst a simple pole at an eigenvalue A.

Remark: If A is not diagonalizable then $(z - A)^{-1}$ is in general no longer has a simple pole at an eigenvalue. For example, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $(z - A)^{-1} = \begin{pmatrix} z^{-1} & z^{-2} \\ 0 & z^{-1} \end{pmatrix}$

$$= z^{-2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Multiple eigenvalues

Suppose $A(\beta)$ depends analytically on β in some region $\Omega \subset \mathbb{C}$. The general question is this: How do the eigenvalues $\{\lambda_i(\beta)\}$ of $A(\beta)$ depend on β?

If $\lambda_i(\beta_0)$ is algebraically simple, we have seen that in a nbhood of β_0, $\lambda_i(\beta)$ is analytic.

But what if $\lambda_i(\beta_0)$ is multiple?

Consider the following example in $\mathcal{H} = \mathbb{C}^6$; here $\Omega = \mathbb{C}$ and $\beta_0 = 0$.

$$A(\beta) = \left(\begin{array}{cc|cc|cc} 0 & \beta & & & & \\ \beta & 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline & & a & \beta & & \\ \multicolumn{2}{c|}{0} & \beta & -a & \multicolumn{2}{c}{0} \\ \hline & & & & b & 1 \\ \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & \beta & b \end{array}\right)$$

where a and b are real non-zero numbers. Then the eigenvalues of $A(\beta)$ are

$$\beta, -\beta, \pm\sqrt{a^2+\beta^2}, \quad b \pm \sqrt{\beta}$$

This example illustrates the following <u>general</u> <u>fact</u>:

the eigenvalues $\lambda_1(\beta), \dots, \lambda_n(\beta)$ of $A(\beta)$ in $\mathbb{C}^n$ split into h cycles, $h \leq n$,

$$\{\lambda_1(\beta), \dots, \lambda_p(\beta)\}, \quad \{\lambda_{p+1}(\beta), \dots, \lambda_{p+q}(\beta)\}, \dots$$

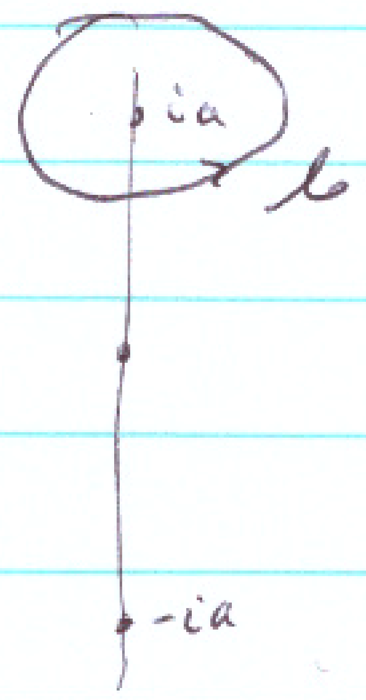
The eigenvalues in each cycle i, $1 \leq i \leq h$, are branches of multiple valued analytic functions f_i, $1 \leq i \leq h$, which have singularities at <u>special</u> <u>points</u>. Moreover, the special points form a discrete set (<u>note</u>: a set S is <u>discrete</u>

if $K \cap S$ is finite for all compact $K \subset \Omega$)

- Here $\lambda_1(\beta) = \beta$ is a cycle with one member and the multiple valued analytic function $f_1(\beta) = \beta$ is, in this case, single valued.

- Similarly, $\lambda_2(\beta) = -\beta$, and $f_2(\beta) = -\beta$ is single valued

- Now $\{\lambda_3(\beta), \lambda_4(\beta)\} = \{\pm\sqrt{a^2+\beta^2}\}$
  Here the cycle has 2 members given by the 2 branches of the multiple valued function $f_3(\beta) = \sqrt{a^2+\beta^2}$. This is analytic in $\mathbb{C} \setminus \{ia, -ia\}$ and the special points are $\pm ia$



Note that if we follow $f_3(\beta) = \sqrt{a^2+\beta^2}$ around the circle b, $\sqrt{a^2+\beta^2} \to -\sqrt{a^2+\beta^2}$ ie $\lambda_3(\beta) \to \lambda_4(\beta)$
Going around b one more time, returns us to $\lambda_3(\beta)$, $\lambda_3(\beta) \to \lambda_4(\beta) \to \lambda_3(\beta)$.
Similarly for a circle around $-ia$.
More precisely, $f_3(\beta) = \sqrt{a^2+\beta^2}$ is a single valued function on an appropriate Riemann surface. As a general fact, if $A(\beta)$ is a polynomial in β, the multivalued analytic function $f_i(\beta)$ will always

be a single valued function on a Riemann surface.

- $\{\lambda_5(\beta), \lambda_6(\beta)\} = \{b \pm \sqrt{\beta}\}$.

Again the cycle has 2 members. They are branches of the multiple valued analytic function $f_4(\beta) = b + \sqrt{\beta}$ and as above passing around the circle C,

$\lambda_5(\beta) \to \lambda_6(\beta)$. Passing a second time, $\lambda_6(\beta) \to \lambda_5(\beta)$.

$\{0\}$ is the set of special points for $f_4(\beta)$.

The multivalued analytic functions whose branches constitute the eigenvalue groups are distinct in the above example, i.e. f_1, f_2, f_3 and f_4 are distinct. But this clearly need not be true in general, e.g. if

$$\tilde{A}(\beta) \left(\begin{array}{cc|cc|cc} a & \beta & & 0 & & 0 \\ \beta & -a & & & & \\ \hline & 0 & a & \beta & & 0 \\ & & \beta & -a & & \\ \hline & 0 & & 0 & b & 1 \\ & & & & \beta & b \end{array}\right), \quad a \neq 0, \ b \neq 0$$

Then the eigenvalues as $\pm\sqrt{a^2+\beta^2}$, $\pm\sqrt{a^2+\beta^2}$, $b \pm \sqrt{\beta}$ so that $h = 3$ and $f_1(\beta) = f_2(\beta) = \sqrt{a^2+\beta^2}$, $f_3(\beta) = b + \sqrt{\beta}$.

<u>More is known</u>: The multivalued functions $f_1(\beta), \ldots, f_n(\beta)$, have at worst <u>algebraic singularities</u> at their special points. This means that if $\hat{\beta}$, say, is a special point of $f_i(\beta)$, then $f_i(\beta)$ has a convergent power series expansion in $(\beta - \hat{\beta})^{\frac{1}{m}}$ for some positive integer m,

$$f_i(\beta) = f_i(\hat{\beta}) + a_1 (\beta - \hat{\beta})^{\frac{1}{m}} + a_2 (\beta - \hat{\beta})^{\frac{2}{m}} + \cdots$$

for $|\beta - \hat{\beta}|$ small. (This is called a <u>Puiseux series</u>).

In particular, as no negative powers of $(\beta - \hat{\beta})^{\frac{1}{m}}$ occur, the functions $f_i(\beta)$ are always continuous at the special points, and hence everywhere.

For example in the example on p162, $f_4(\beta) = b + \sqrt{\beta}$, so that $m = 2$ in this case. Similarly, for $f_3(\beta) = \sqrt{a^2 + \beta^2}$ $= \sqrt{\beta - ia}\,\sqrt{\beta + ia} = \sqrt{\beta - ia}\,\sqrt{(\beta - ia) + 2a}$, and again $m = 2$ at the special point $\hat{\beta} = ia$ (and similarly $\hat{\beta} = -ia$).

A general reference for the nature of the eigenvalues is Knopp's book on analytic functions, Vol III.